

Inverse Kinematics

Inverse Kinematics

- ▶ given the pose of the end effector, find the joint variables that produce the end effector pose
- ▶ for a 6-joint robot, given

$$T^0_6 = \begin{bmatrix} R^0_6 & o^0_6 \\ \mathbf{0} & 1 \end{bmatrix}$$

find

$$q_1, q_2, q_3, q_4, q_5, q_6$$

RPP + Spherical Wrist

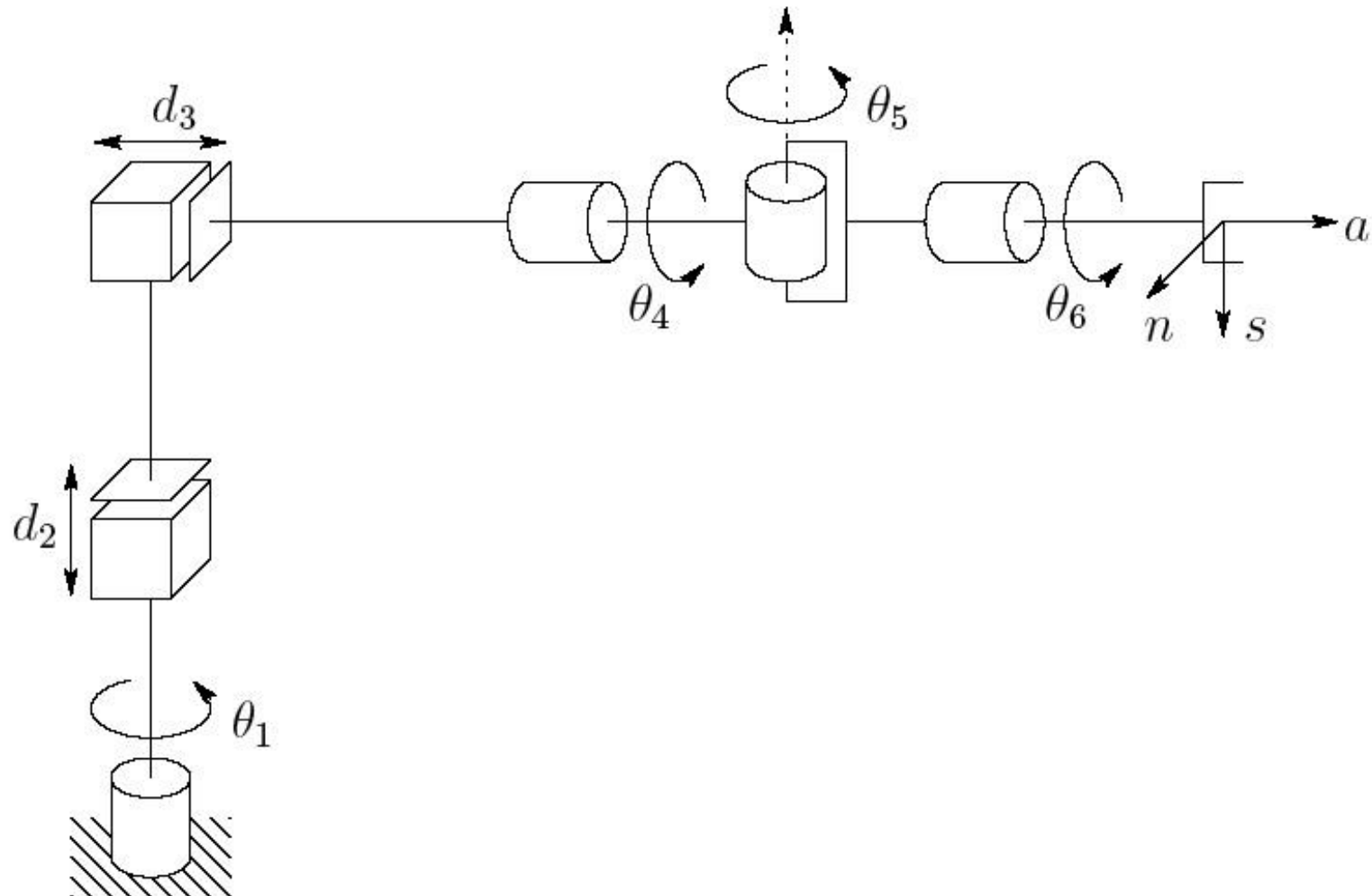


Figure 3.9: Cylindrical robot with spherical wrist.

RPP + Spherical Wrist

- ▶ solving for the joint variables directly is hard

$$T_6^0 = T_3^0 T_6^3 = \begin{bmatrix} r_{11} & r_{12} & r_{13} & d_x \\ r_{21} & r_{22} & r_{23} & d_y \\ r_{31} & r_{32} & r_{33} & d_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$r_{11} = c_1 c_4 c_5 c_6 - c_1 s_4 s_6 + s_1 s_5 c_6$$

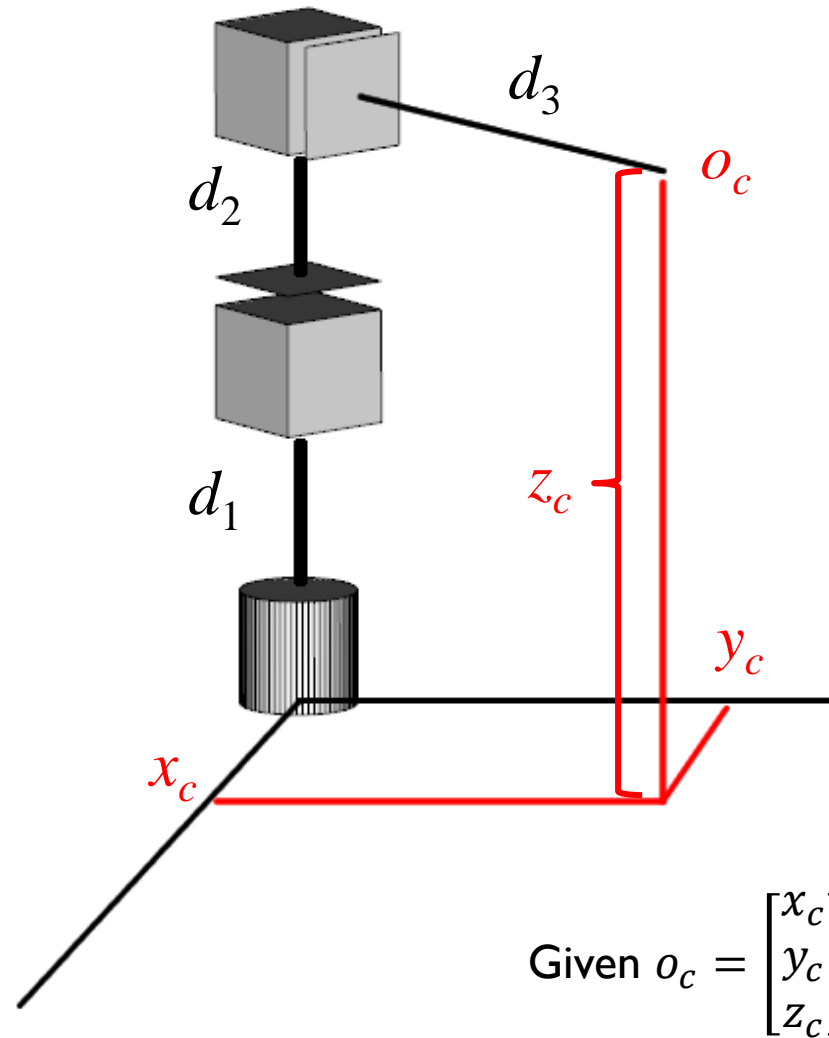
$$\vdots$$

$$d_z = -s_4 s_5 d_6 + d_1 + d_2$$

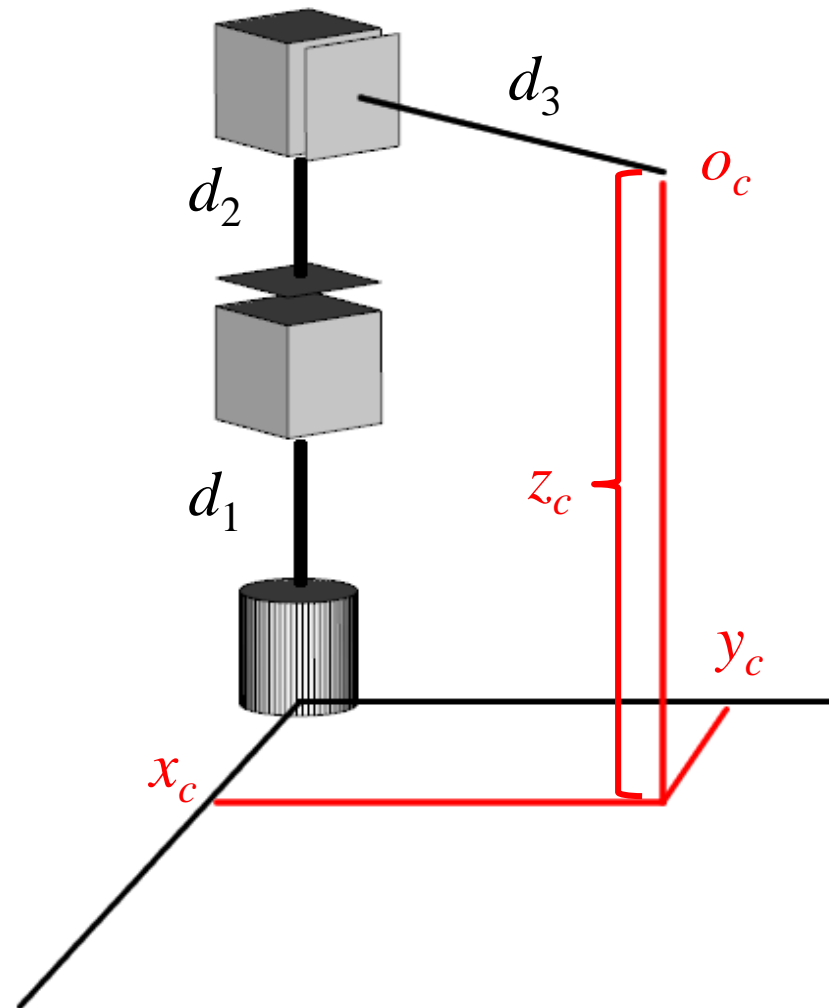
Kinematic Decoupling

- ▶ for 6-joint robots where the last 3 joints intersecting at a point (e.g., last 3 joints are spherical wrist) there is a simpler way to solve the inverse kinematics problem
 1. use the intersection point (wrist center) to solve for the first 3 joint variables
 - ▶ inverse position kinematics
 2. use the end-effector pose to solve for the last 3 joint variables
 - ▶ inverse orientation kinematics

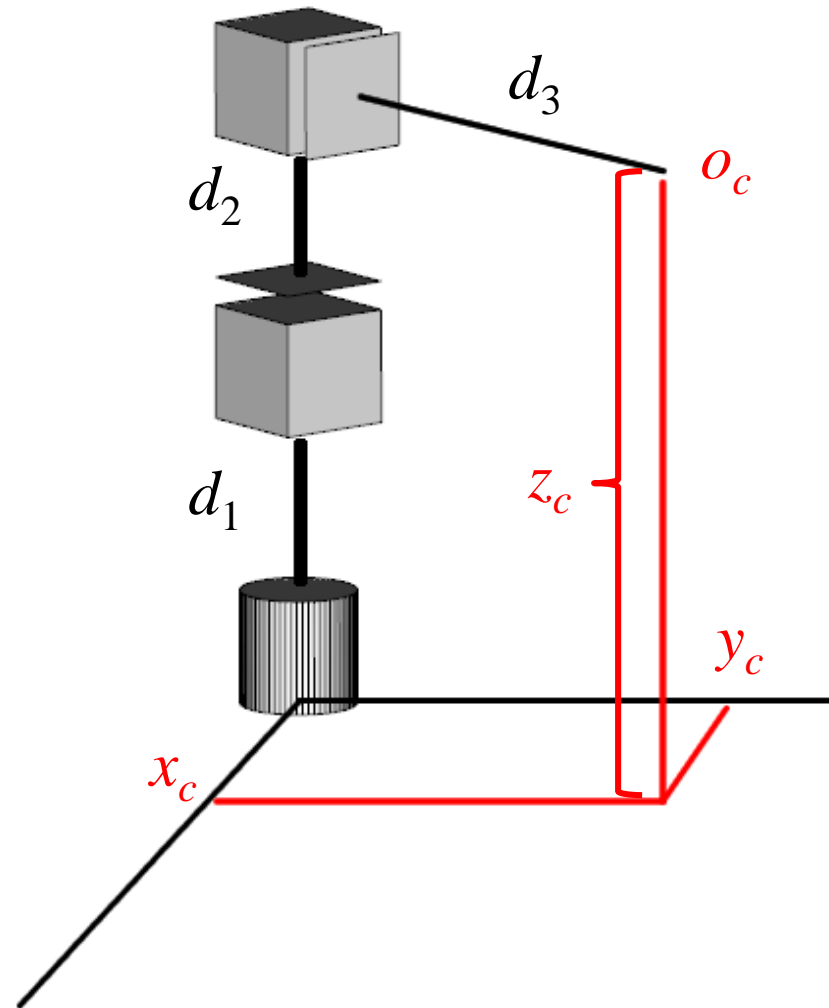
RPP Cylindrical Manipulator



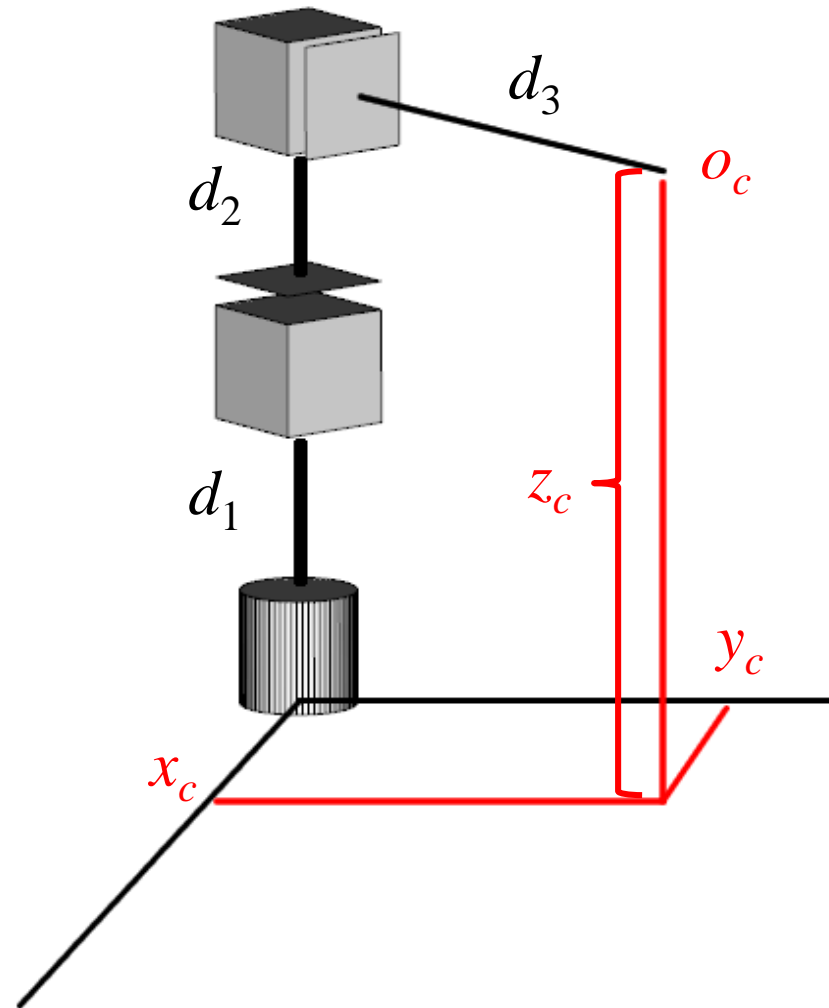
RPP Cylindrical Manipulator



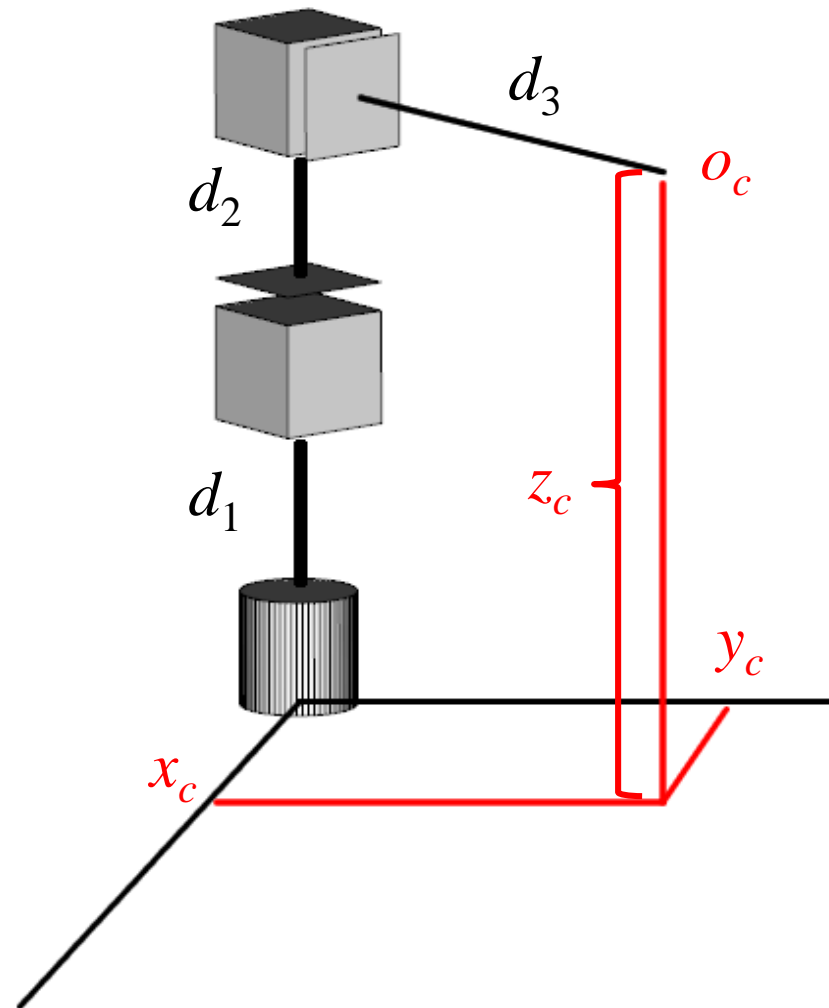
RPP Cylindrical Manipulator



RPP Cylindrical Manipulator



RPP Cylindrical Manipulator



RRP Spherical Manipulator

Given $o_c = \begin{bmatrix} x_c \\ y_c \\ z_c \end{bmatrix}$ find θ_1, θ_2, d_3

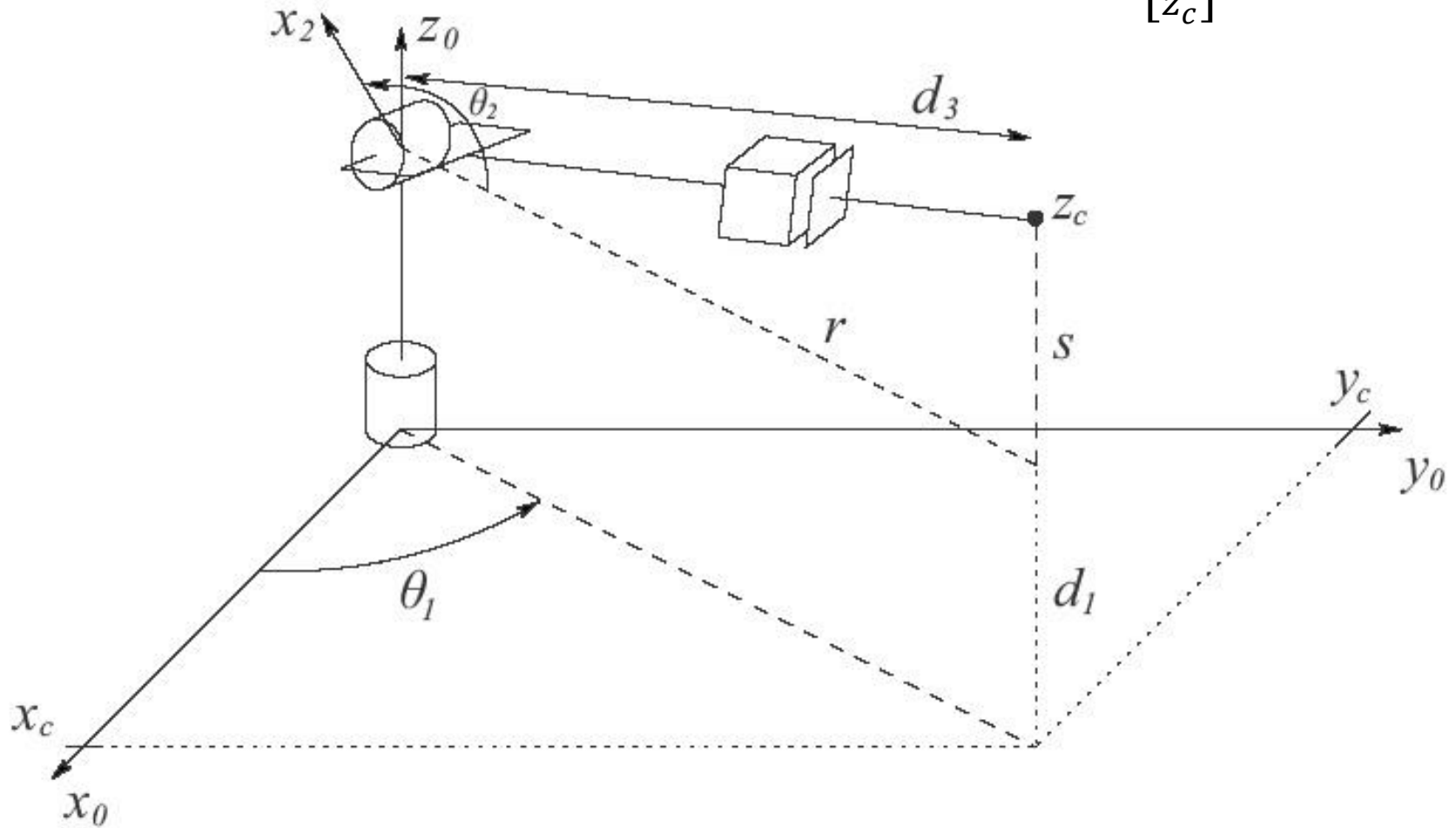


Figure 3.21: Spherical manipulator.

RRP Spherical Manipulator

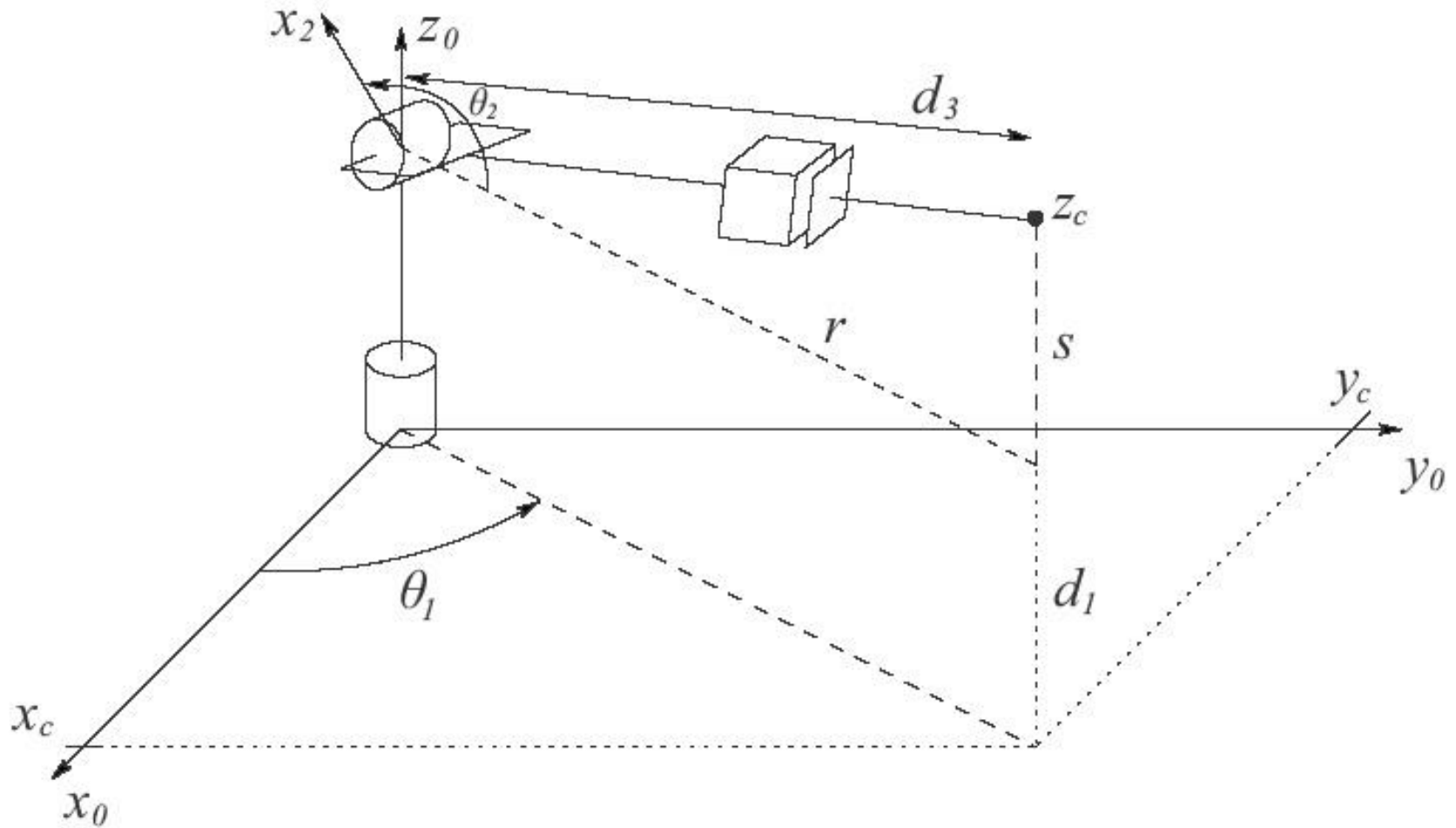


Figure 3.21: Spherical manipulator.

RRP Spherical Manipulator

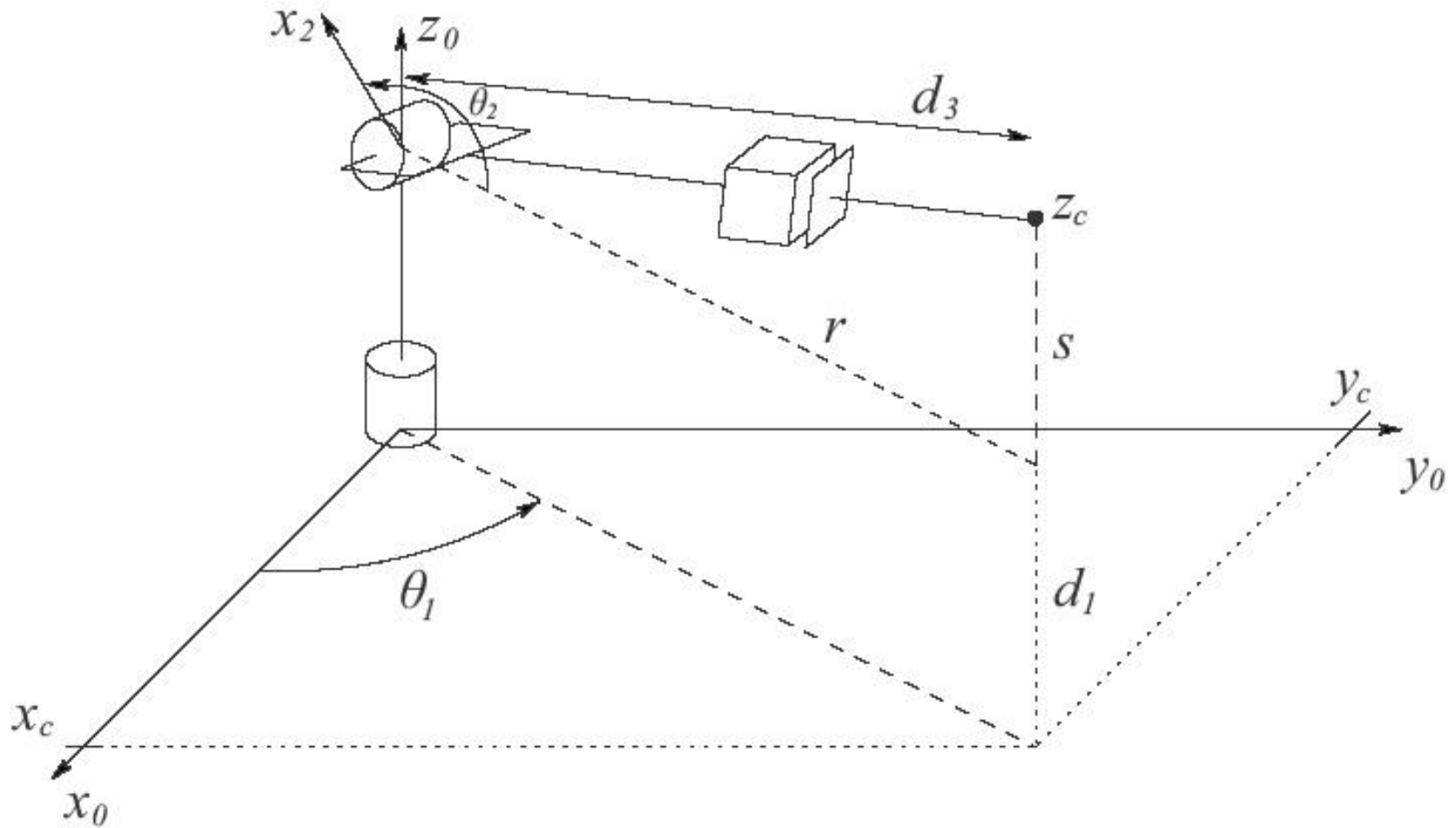


Figure 3.21: Spherical manipulator.

RRP Spherical Manipulator

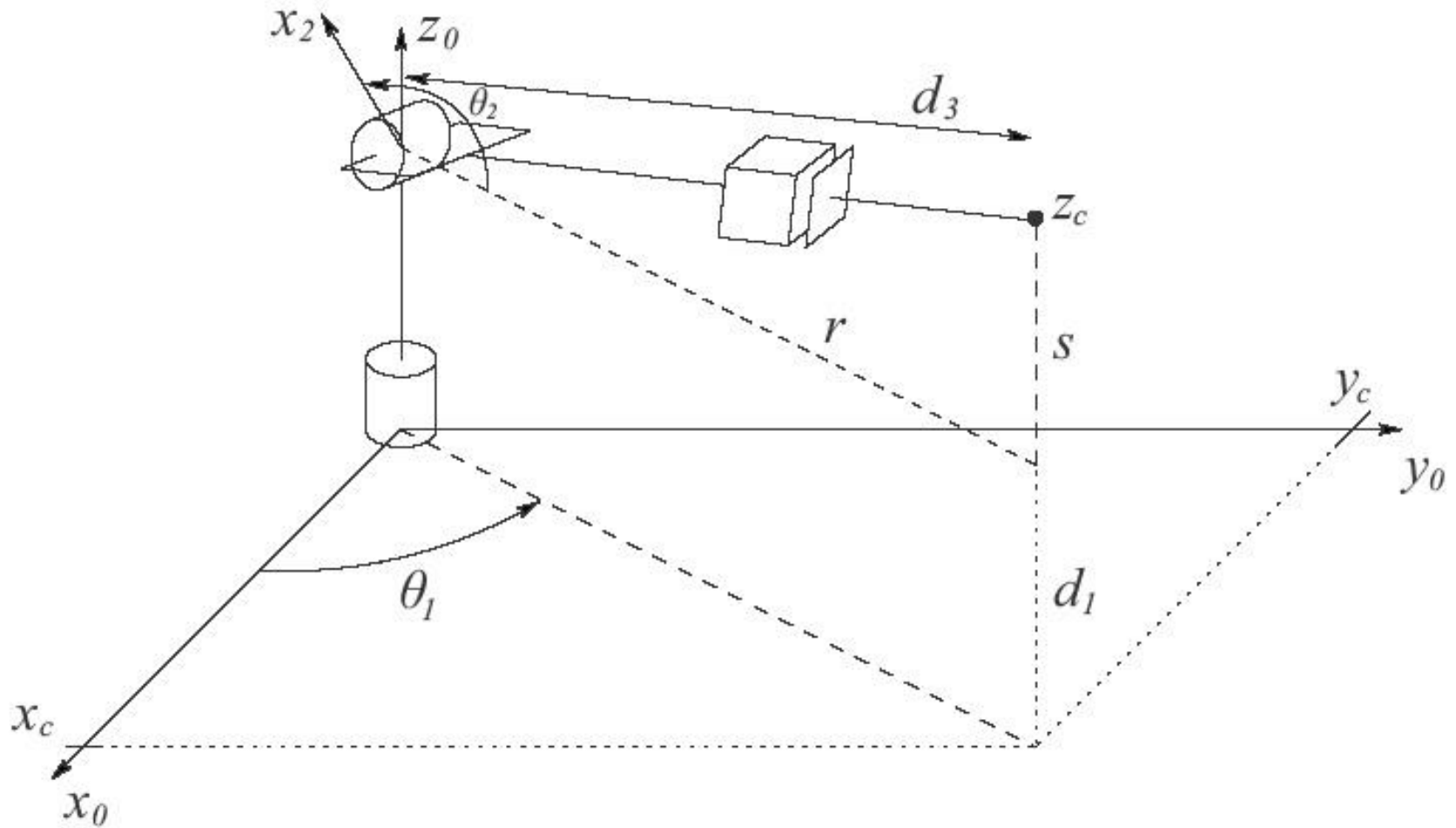


Figure 3.21: Spherical manipulator.

RRP Spherical Manipulator

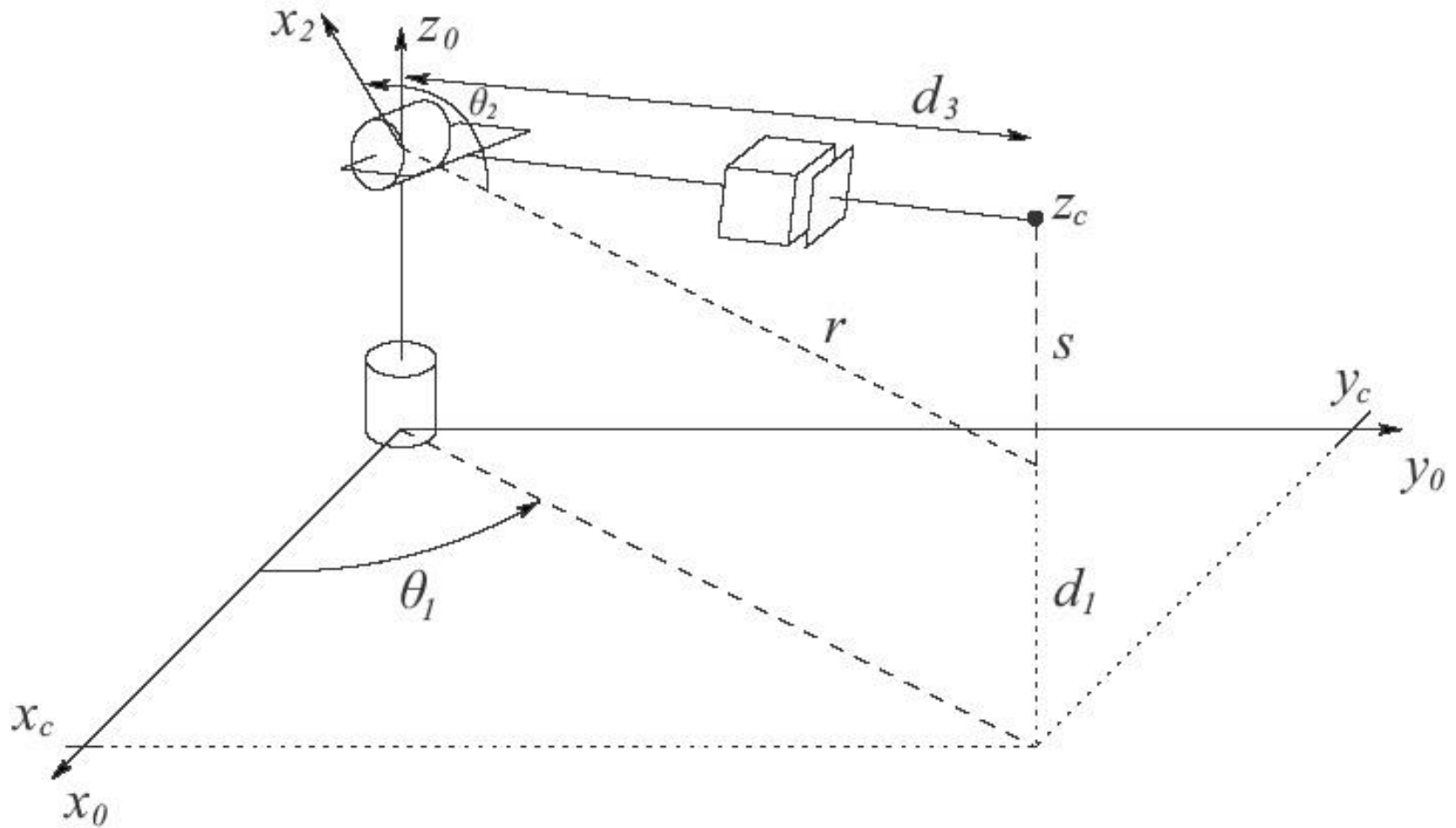
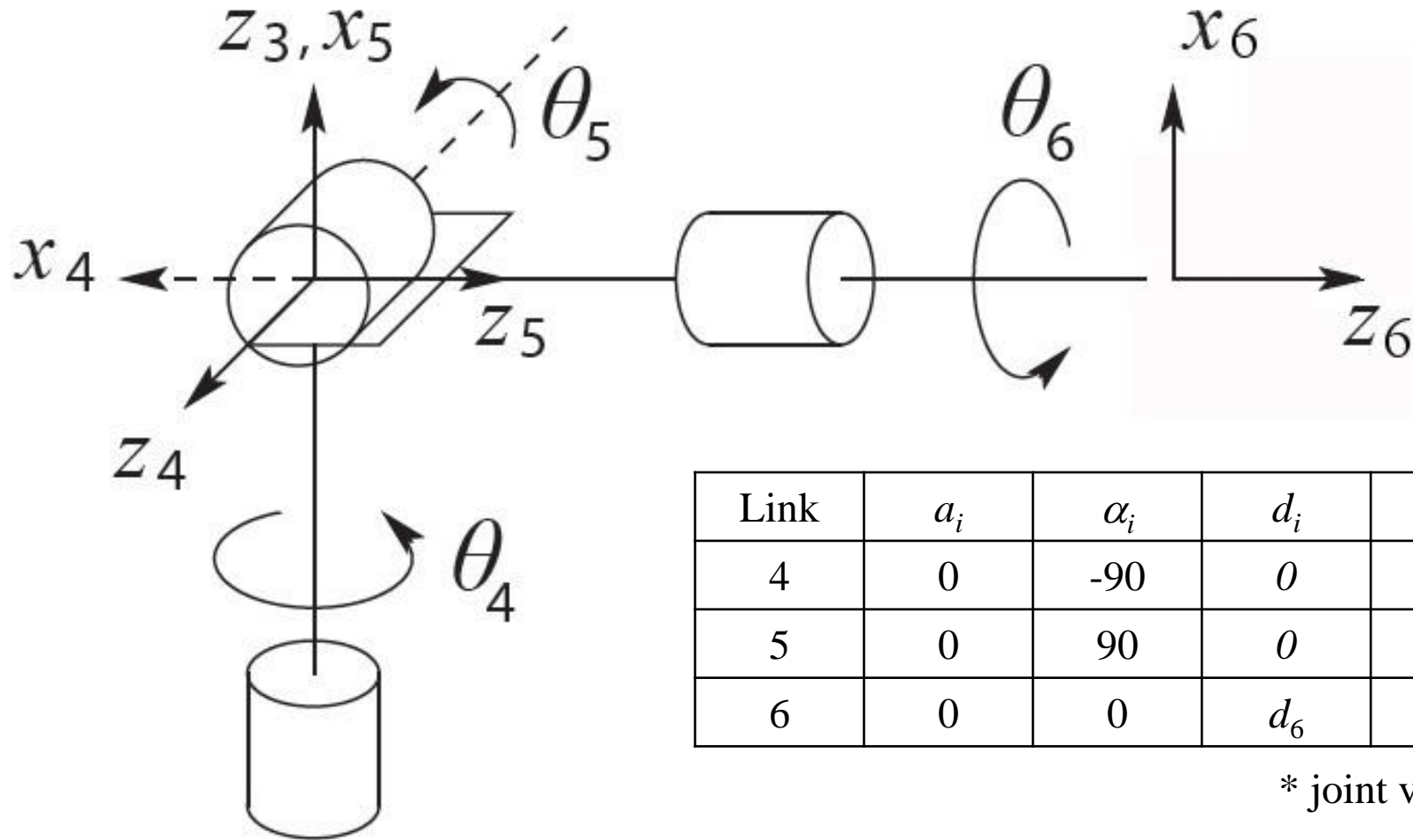


Figure 3.21: Spherical manipulator.

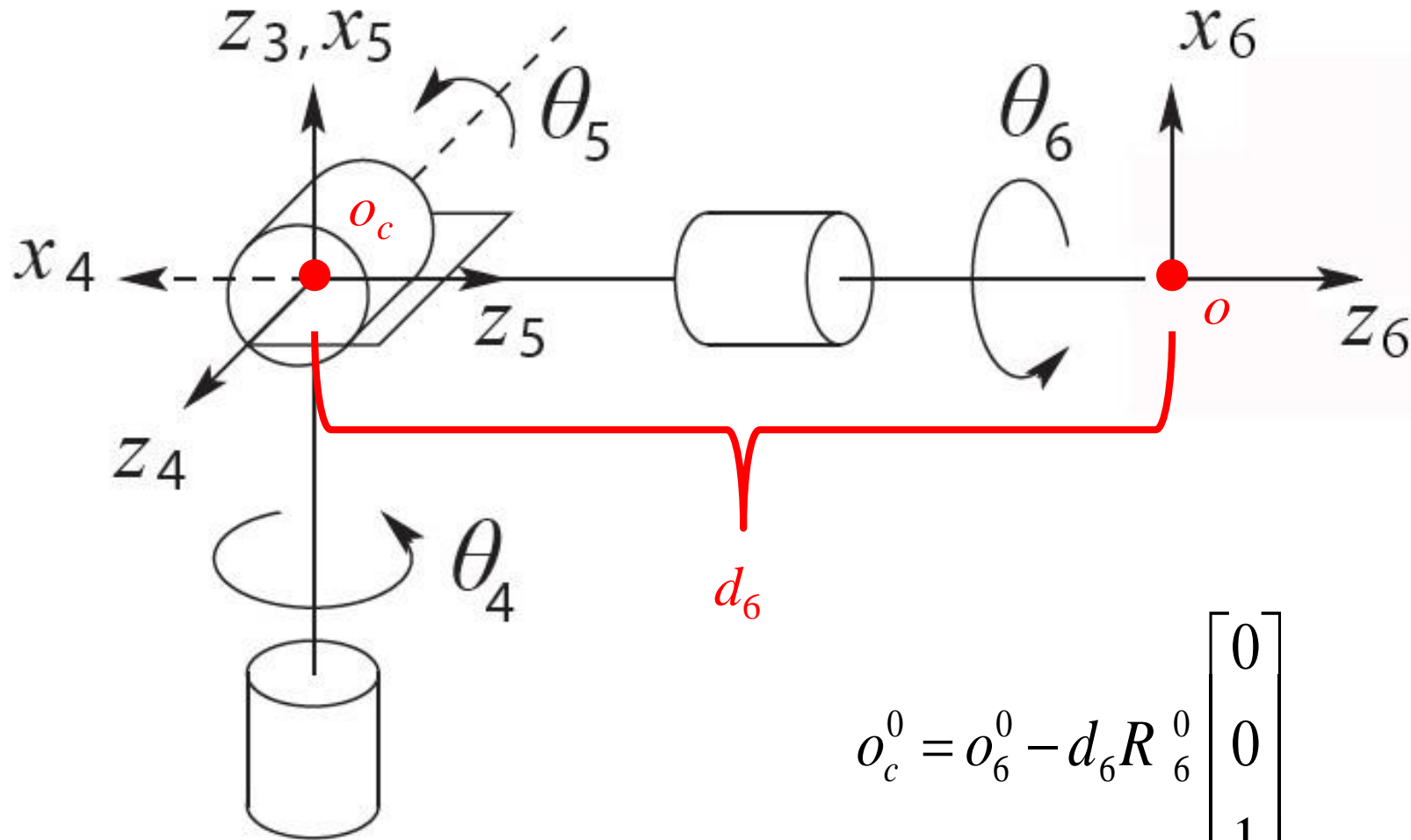
Spherical Wrist



Spherical Wrist

$$T_6^3 = T_4^3 T_5^4 T_6^5 = \begin{bmatrix} c_4 c_5 c_6 - s_4 s_6 & -c_4 c_5 s_6 - s_4 c_6 & c_4 s_5 & c_4 s_5 d_6 \\ s_4 c_5 c_6 + c_4 s_6 & -s_4 c_5 s_6 + c_4 c_6 & s_4 s_5 & s_4 s_5 d_6 \\ -s_5 c_6 & s_5 s_6 & c_5 & c_5 d_6 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Spherical Wrist



$$o_c^0 = o_6^0 - d_6 R_6^0 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Inverse Kinematics Recap

1. Solve for the first 3 joint variables q_1, q_2, q_3 such that the wrist center o_c has coordinates

$$o_c^0 = o_6^0 - d_6 R_6^0 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

2. Using the results from Step 1, compute R_3^0
3. Solve for the wrist joint variables q_4, q_5, q_6 corresponding to the rotation matrix

$$R_6^3 = \left(R_3^0 \right)^T R_6^0$$

Spherical Wrist

► for the spherical wrist

$$T_6^3 = T_4^3 T_5^4 T_6^5 = \begin{bmatrix} c_4 c_5 c_6 - s_4 s_6 & -c_4 c_5 s_6 - s_4 c_6 & c_4 s_5 & c_4 s_5 d_6 \\ s_4 c_5 c_6 + c_4 s_6 & -s_4 c_5 s_6 + c_4 c_6 & s_4 s_5 & s_4 s_5 d_6 \\ -s_5 c_6 & s_5 s_6 & c_5 & c_5 d_6 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

if $s_5 \neq 0$

$$\theta_5^{\text{pos}} = \text{atan2} \left(\sqrt{1 - r_{33}^2}, r_{33} \right)$$

$$\theta_5^{\text{neg}} = \text{atan2} \left(-\sqrt{1 - r_{33}^2}, r_{33} \right)$$

Spherical Wrist

$$T_6^3 = T_4^3 T_5^4 T_6^5 = \begin{bmatrix} c_4 c_5 c_6 - s_4 s_6 & -c_4 c_5 s_6 - s_4 c_6 & c_4 s_5 & c_4 s_5 d_6 \\ s_4 c_5 c_6 + c_4 s_6 & -s_4 c_5 s_6 + c_4 c_6 & s_4 s_5 & s_4 s_5 d_6 \\ -s_5 c_6 & s_5 s_6 & c_5 & c_5 d_6 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

for $\theta_5^{\text{pos}}, s_5 > 0$

$$\theta_4 = \text{atan2}(r_{23}, r_{13})$$

$$\theta_6 = \text{atan2}(r_{32}, -r_{31})$$

Spherical Wrist

$$T_6^3 = T_4^3 T_5^4 T_6^5 = \begin{bmatrix} c_4 c_5 c_6 - s_4 s_6 & -c_4 c_5 s_6 - s_4 c_6 & c_4 s_5 & c_4 s_5 d_6 \\ s_4 c_5 c_6 + c_4 s_6 & -s_4 c_5 s_6 + c_4 c_6 & s_4 s_5 & s_4 s_5 d_6 \\ -s_5 c_6 & s_5 s_6 & c_5 & c_5 d_6 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

for $\theta_5^{\text{neg}}, s_5 < 0$

$$\theta_4 = \text{atan2}(-r_{23}, -r_{13})$$

$$\theta_6 = \text{atan2}(-r_{32}, r_{31})$$

Spherical Wrist

► if $\theta_5 = 0$

$$\begin{aligned}
 T_6^3 &= T_4^3 T_5^4 T_6^5 = \begin{bmatrix} c_4 c_5 c_6 - s_4 s_6 & -c_4 c_5 s_6 - s_4 c_6 & c_4 s_5 & c_4 s_5 d_6 \\ s_4 c_5 c_6 + c_4 s_6 & -s_4 c_5 s_6 + c_4 c_6 & s_4 s_5 & s_4 s_5 d_6 \\ -s_5 c_6 & s_5 s_6 & c_5 & c_5 d_6 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\
 &= \begin{bmatrix} c_4 c_6 - s_4 s_6 & -c_4 s_6 - s_4 c_6 & 0 & 0 \\ s_4 c_6 + c_4 s_6 & -s_4 s_6 + c_4 c_6 & 0 & 0 \\ 0 & 0 & 1 & d_6 \\ 0 & 0 & 0 & 1 \end{bmatrix}
 \end{aligned}$$

Spherical Wrist

- ▶ continued from previous slide

$$= \begin{bmatrix} c_4 c_6 - s_4 s_6 & -c_4 s_6 - s_4 c_6 & 0 & 0 \\ s_4 c_6 + c_4 s_6 & -s_4 s_6 + c_4 c_6 & 0 & 0 \\ 0 & 0 & 1 & d_6 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} c_{4+6} & -s_{4+6} & 0 & 0 \\ s_{4+6} & c_{4+6} & 0 & 0 \\ 0 & 0 & 1 & d_6 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

only the sum $\theta_4 + \theta_6$
can be determined

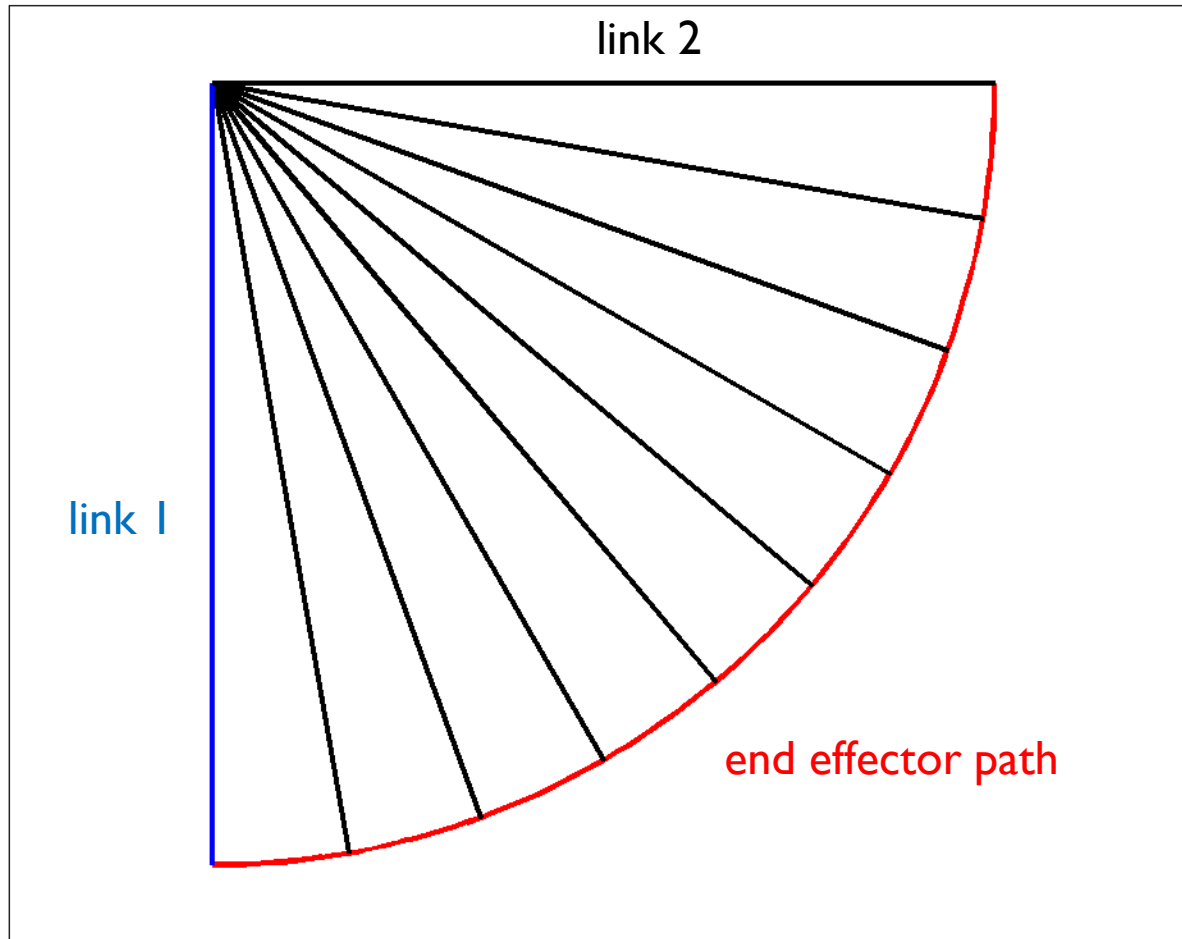
Using Inverse Kinematics in Path Generation

Path Generation

- ▶ a path is defined as a sequence of configurations a robot makes to go from one place to another
- ▶ a trajectory is a path where the velocity and acceleration along the path also matter

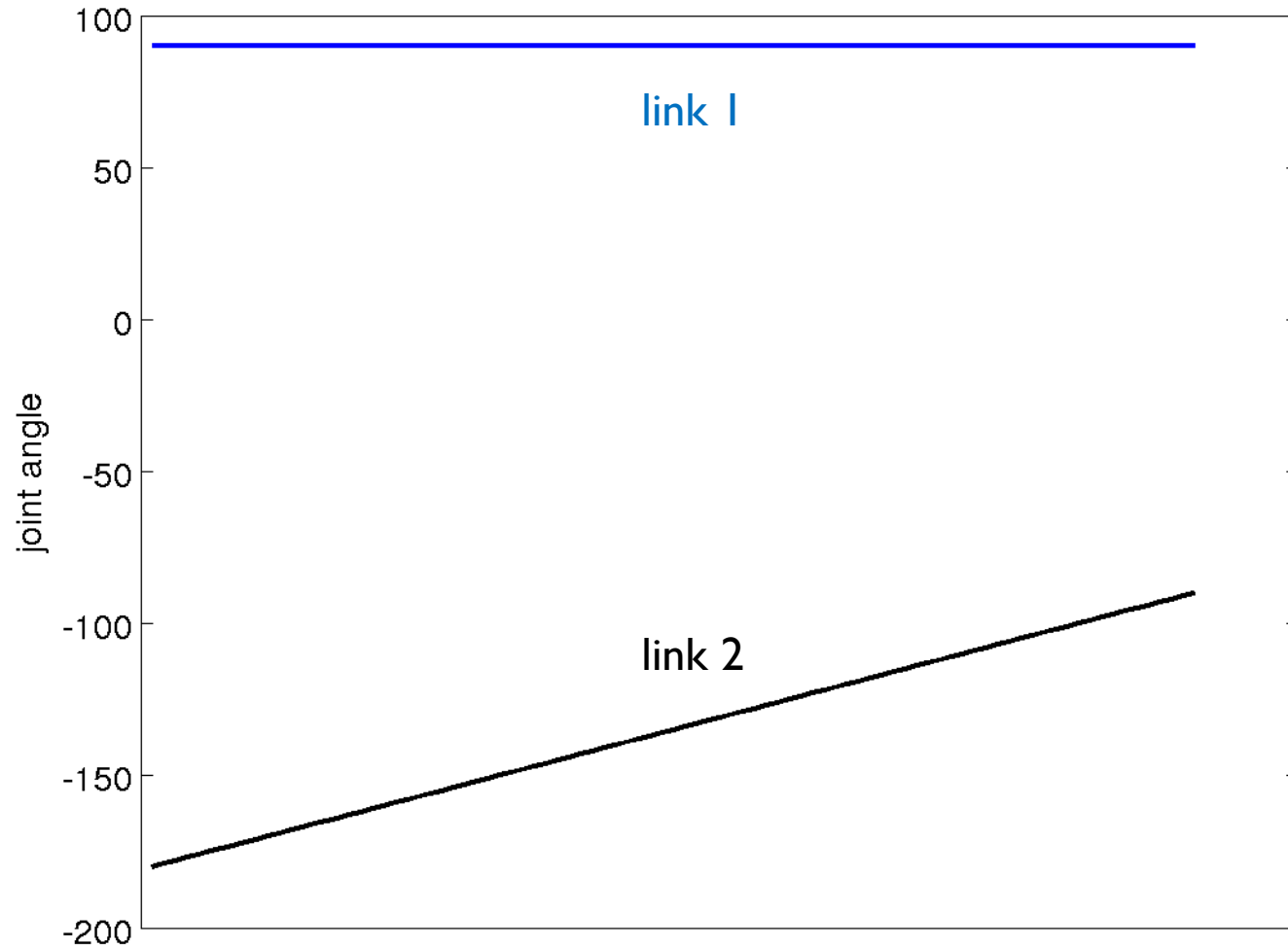
Joint-Space Path

- ▶ a joint-space path is computed considering the joint variables



Joint-Space Path Joint Angles

► linear joint-space path



Joint-Space Path

- ▶ given the current end-effector pose

$0T$

and the desired final end-effector pose

fT

find a sequence of joint angles that generates the path between the two poses

- ▶ idea
 - ▶ solve for the inverse kinematics for the current and final pose to get the joint angles for the current and final pose
 - ▶ interpolate the joint angles

Joint-Space Path

$${}^0T \Rightarrow \text{inverse kinematics} \Rightarrow {}^0Q = \begin{bmatrix} q_1 \\ q_2 \\ \vdots \\ q_n \end{bmatrix}$$

$${}^fT \Rightarrow \text{inverse kinematics} \Rightarrow {}^fQ = \begin{bmatrix} q_1 \\ q_2 \\ \vdots \\ q_n \end{bmatrix}$$

Joint-Space Path

find 0Q from 0T

find fQ from fT

$$\Delta t = 1 / m$$

$$\Delta Q = {}^fQ - {}^0Q$$

for $j = 1$ to m

$$t_j = j \Delta t$$

$${}^jQ = {}^0Q + t_j \Delta Q$$

set joints to jQ

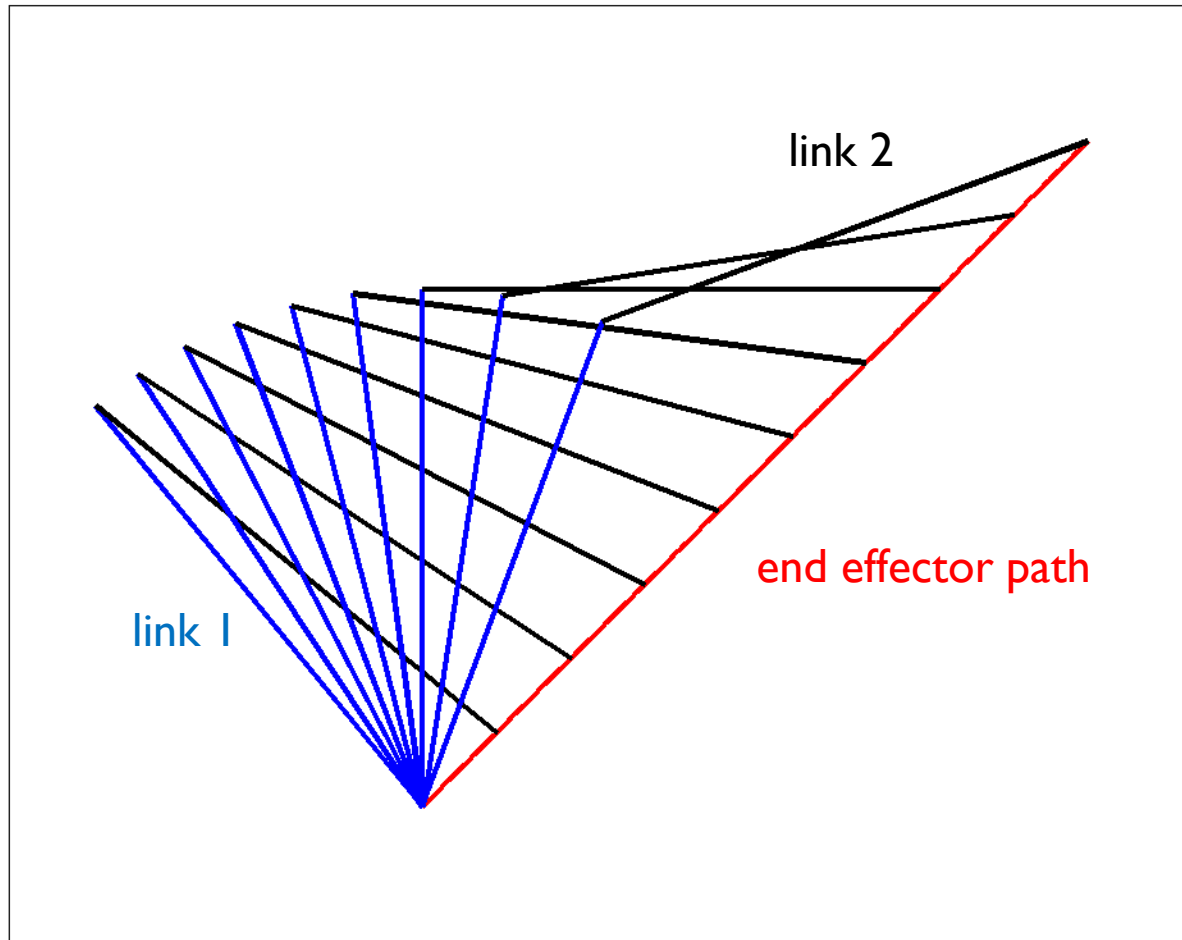
end

Joint-Space Path

- ▶ linearly interpolating the joint variables produces
 - ▶ a linear joint-space path
 - ▶ a non-linear Cartesian path
- ▶ depending on the kinematic structure the Cartesian path can be very complicated
 - ▶ some applications might benefit from a simple, or well defined, Cartesian path

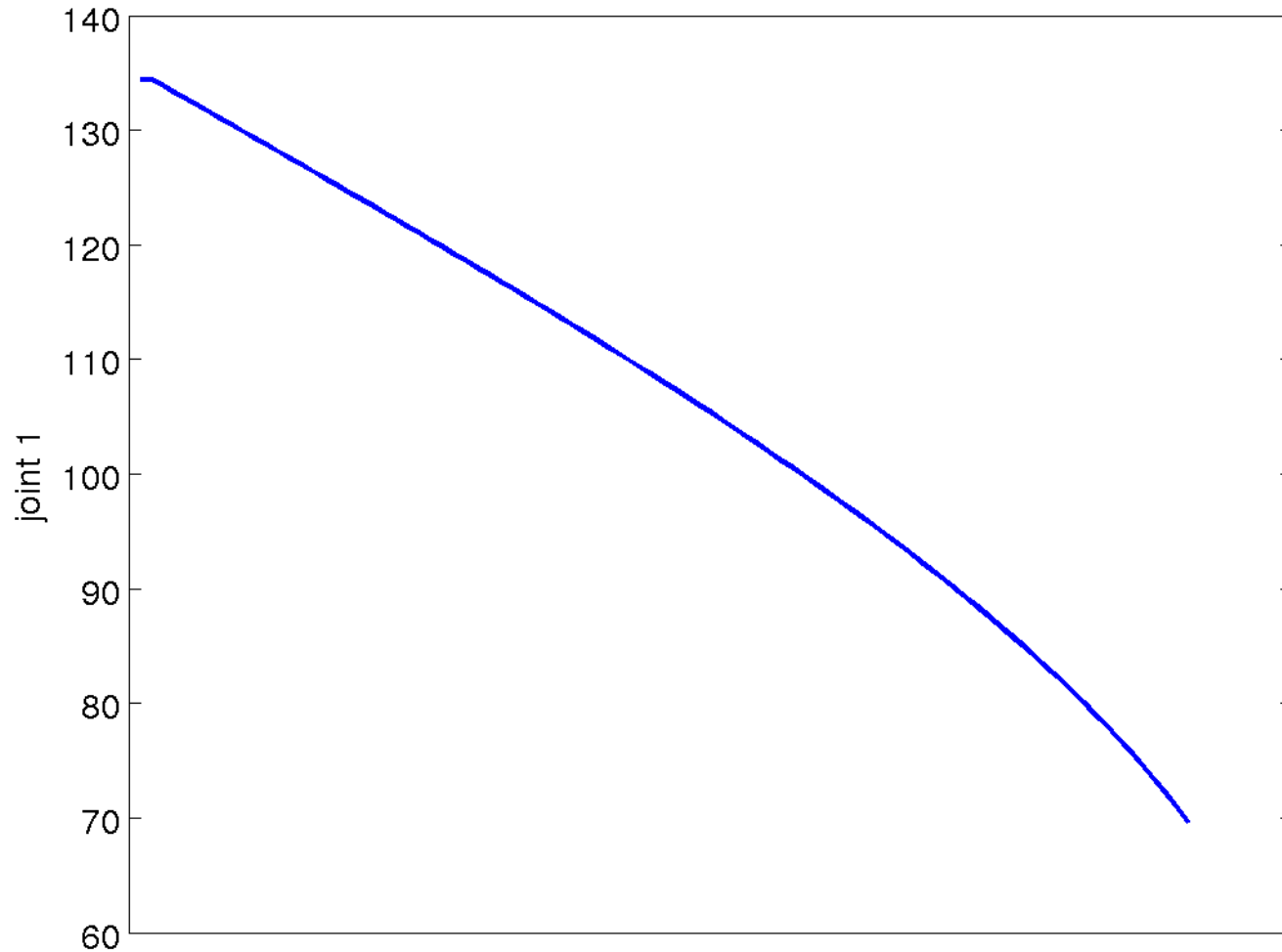
Cartesian-Space Path

- ▶ a Cartesian-space path considers the position of end-effector



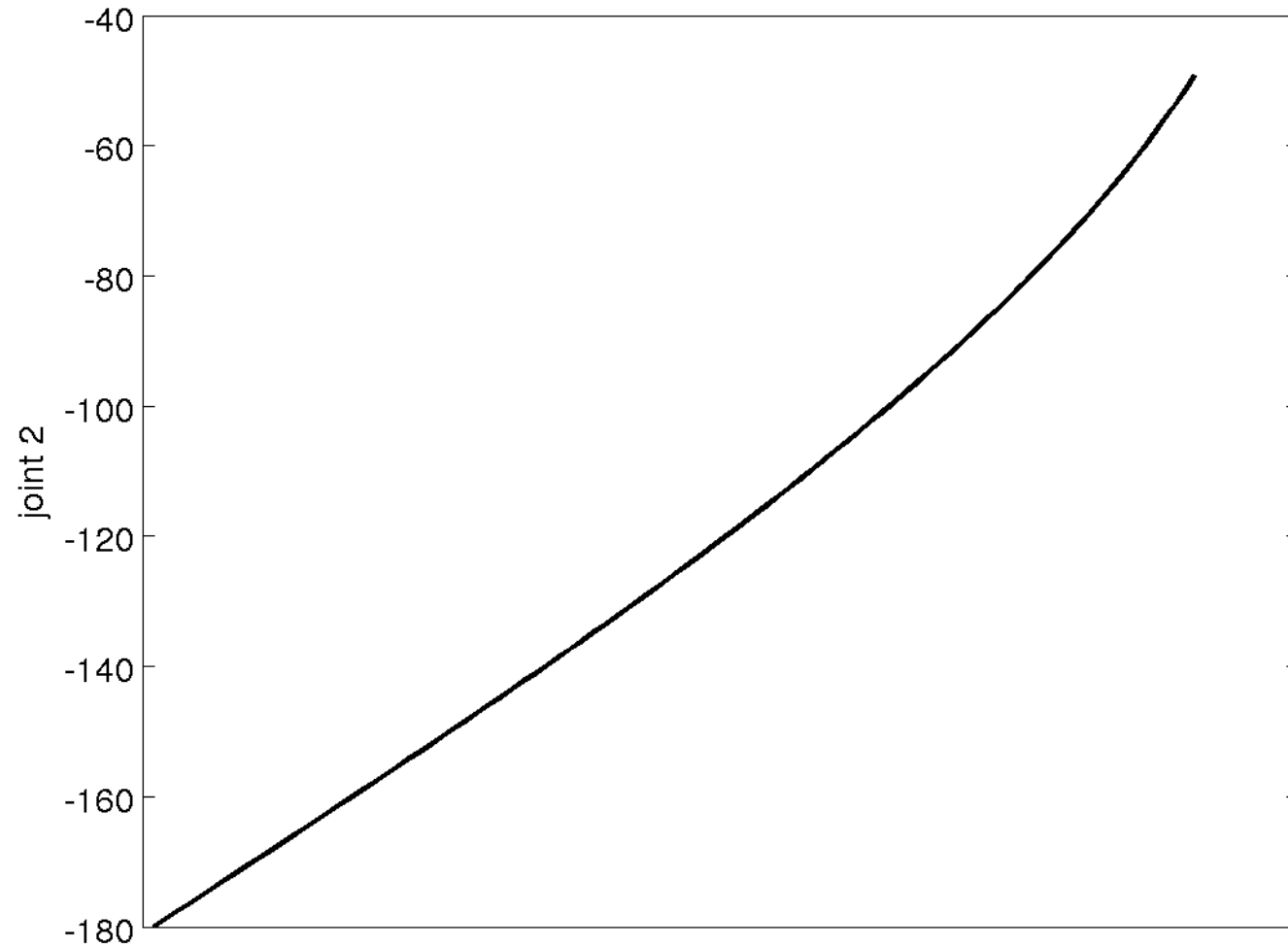
Cartesian-Space Path Joint Variable 1

- ▶ non-linear joint-space path



Cartesian-Space Path Joint Variable 2

► non-linear joint-space path



Issues with Cartesian-Space Paths

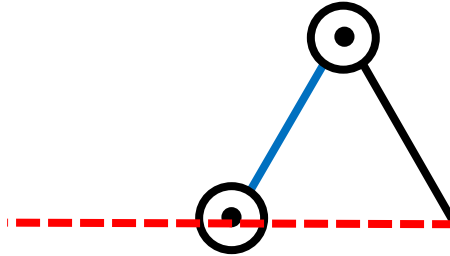
Joint Velocity Issues

- ▶ consider the RR robot shown below
- ▶ assume that the second joint can rotate by ± 180 degrees

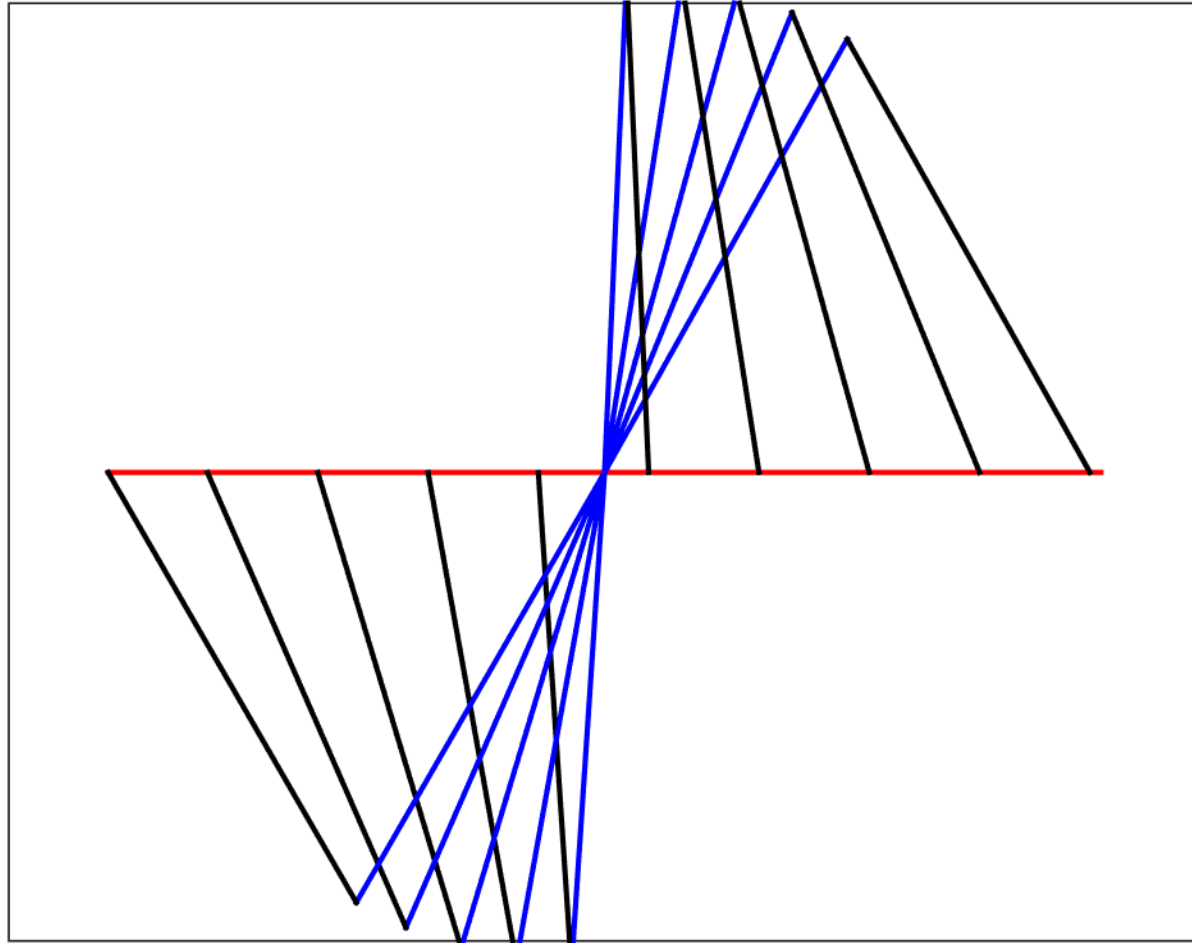


Joint Velocity Issues

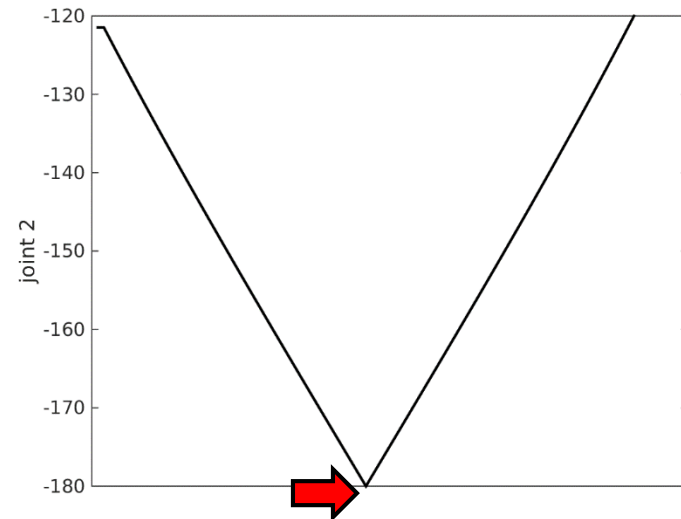
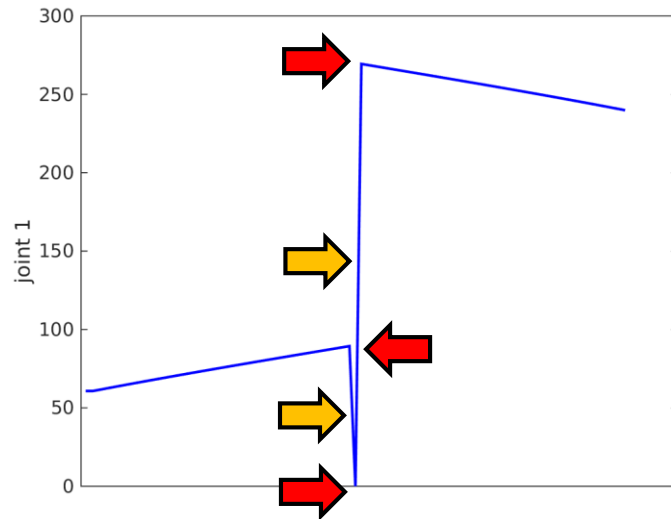
- ▶ what happens when it is commanded to follow the straight line path shown in red?



Joint Velocity Issues



Joint Velocity Issues



jump discontinuity in first derivative = infinite rotational acceleration

steep slope = high rotational velocity

Workspace

- ▶ the *reachable workspace* of a robot is the volume swept by the end effector for all possible combinations of joint variables
 - ▶ i.e., it is the set of all points that the end effector can be moved to

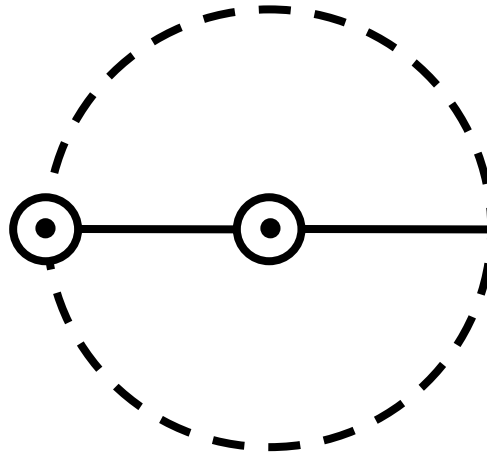
Workspace

- ▶ consider the RR robot shown below
- ▶ assume both joints can rotate by 360 degrees



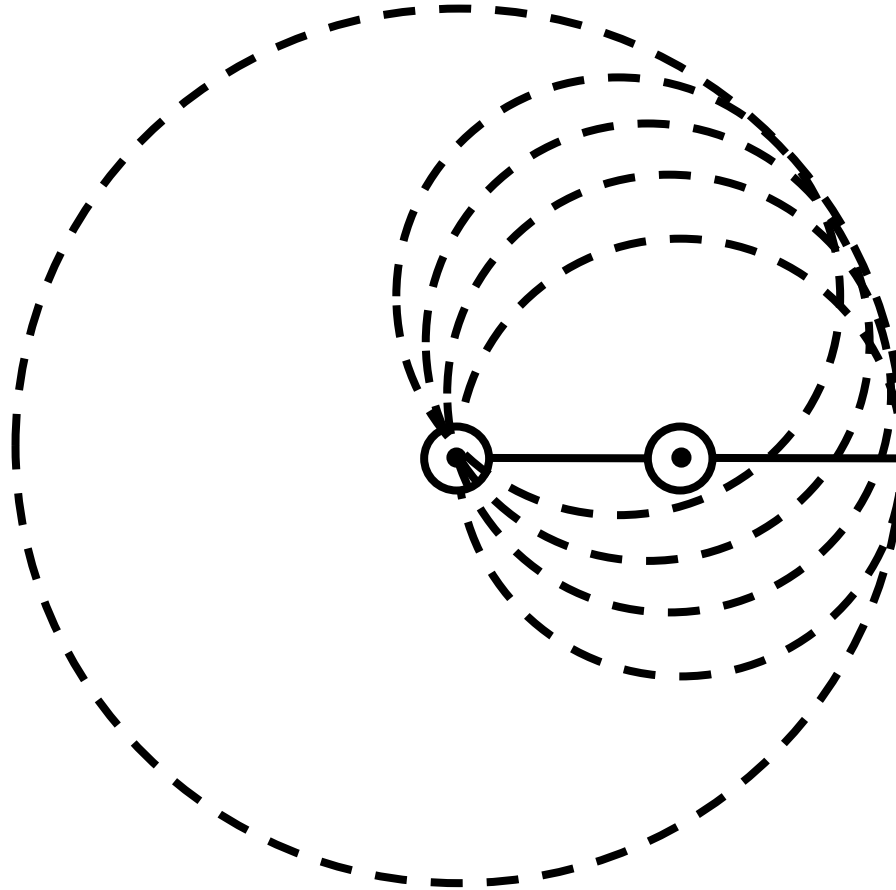
Workspace

- ▶ rotating the second joint through 360 degrees sweeps out the set of points on the dashed circle



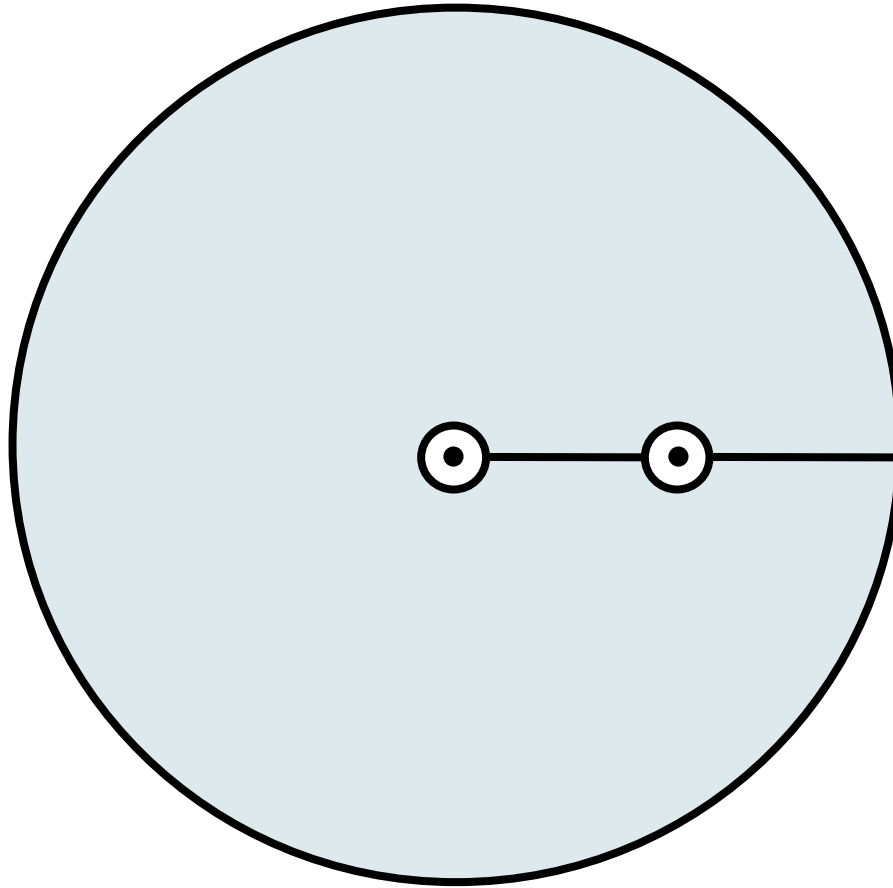
Workspace

- ▶ rotating the first and second joints through 360 degrees sweeps out the set of all points inside the outer dashed circle



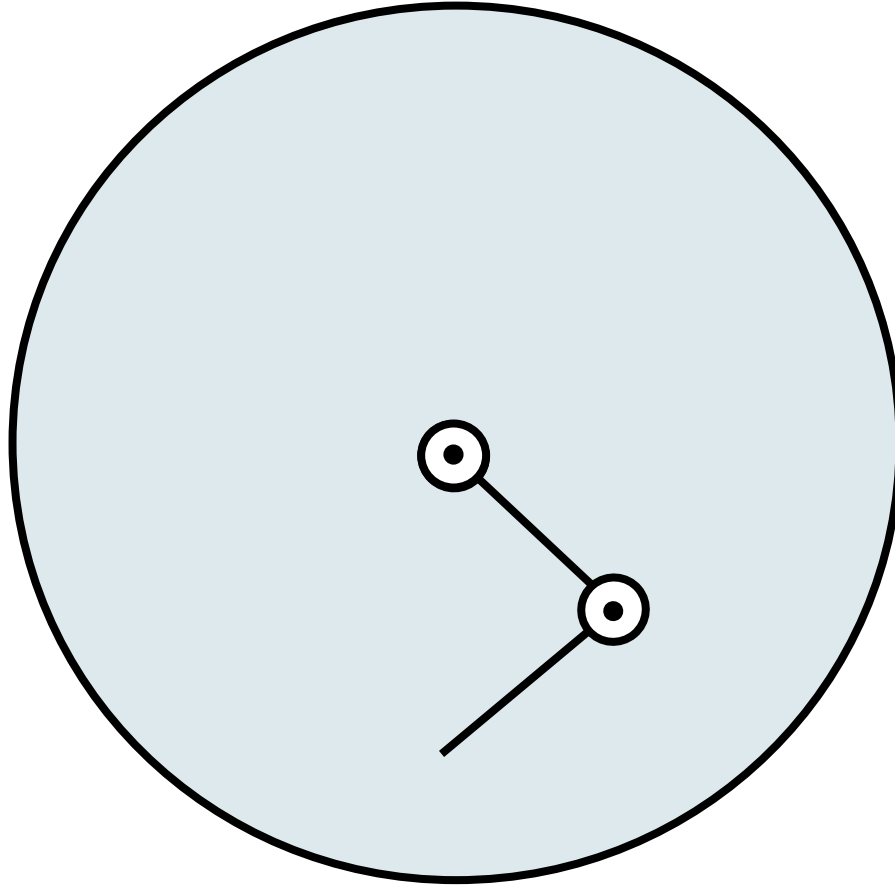
Workspace

- ▶ workspace consists of all of the points inside the gray circle



Workspace

- ▶ workspace consists of all of the points inside the gray circle



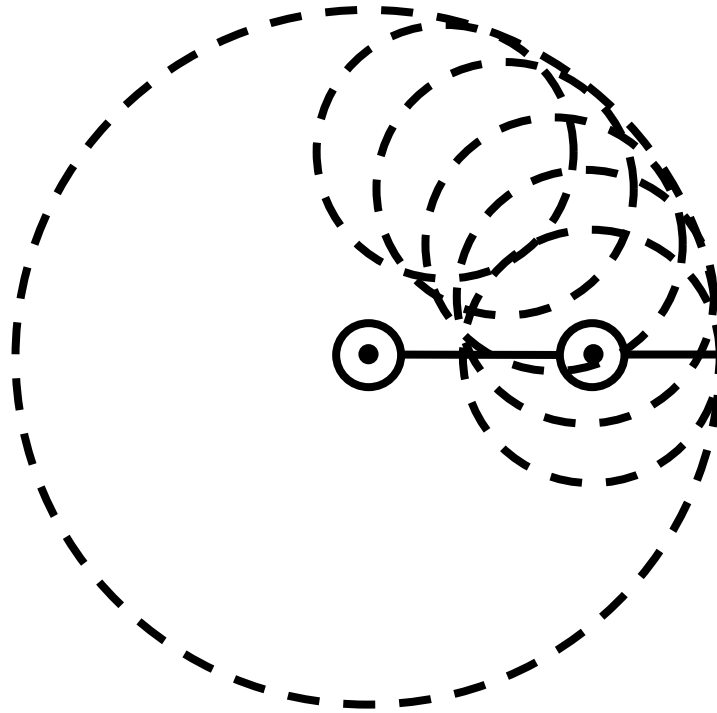
Workspace

- ▶ consider the RR robot shown below where the second link is shorter than the first
- ▶ assume both joints can rotate by 360 degrees



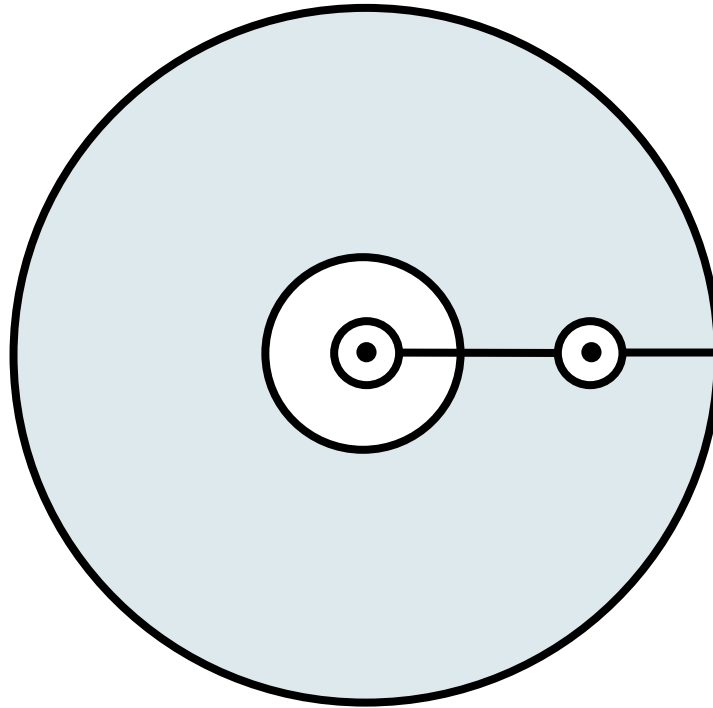
Workspace

- ▶ rotating the second joint through 360 degrees sweeps out the set of points on the dashed circle



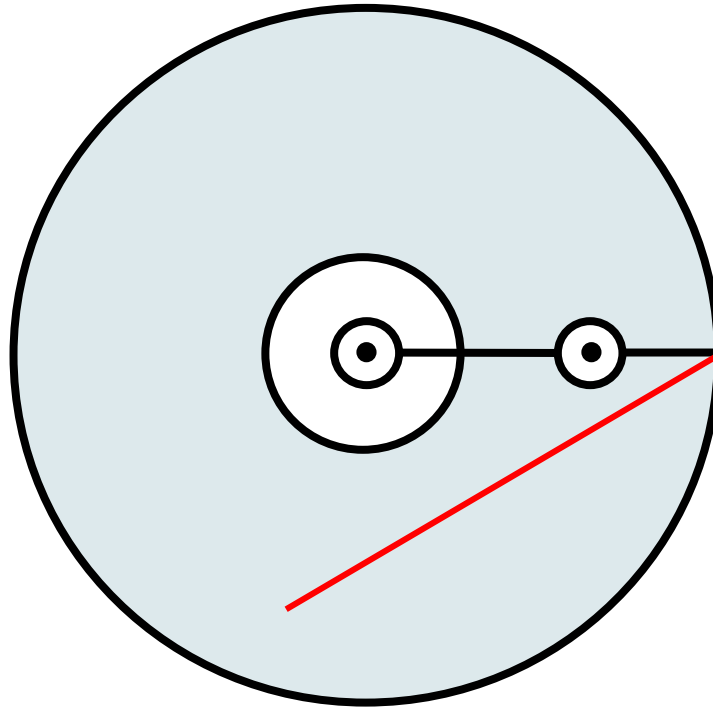
Workspace

- ▶ workspace consists of all of the points inside the gray area



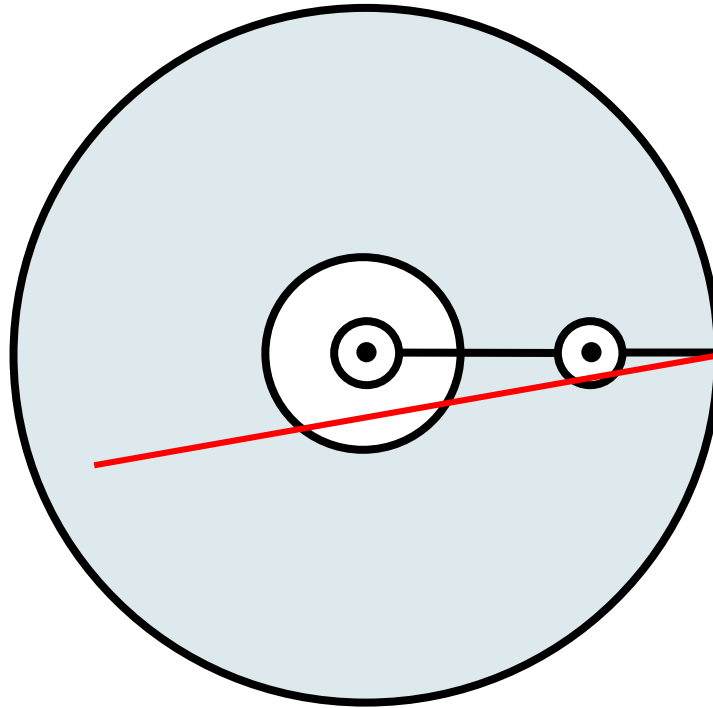
Workspace

- ▶ consider the following straight line path shown in red
- ▶ start point, end point, and all points in between are reachable



Workspace

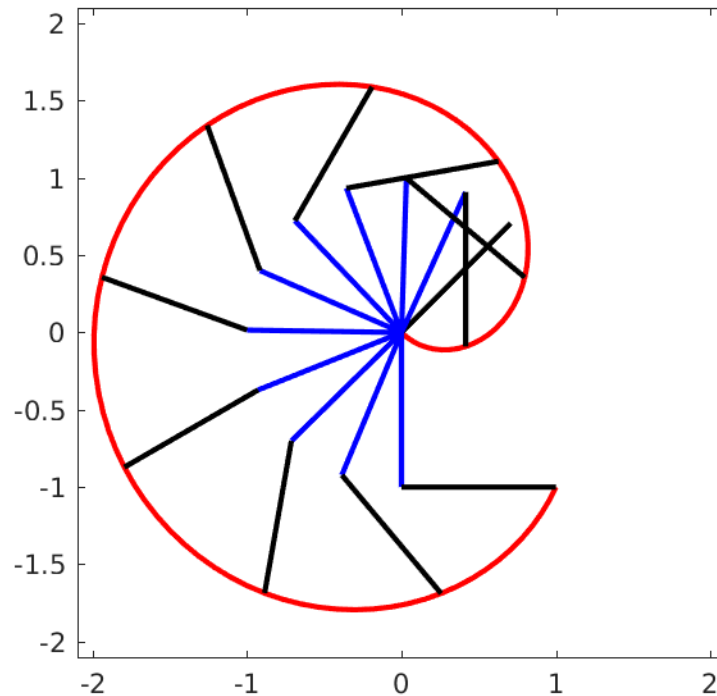
- ▶ consider the following straight line path shown in red
- ▶ start point and end point are reachable, but some points in between are not reachable



Paths satisfying end point constraints

Joint-Space Path

- ▶ a joint-space path is computed considering the joint variables



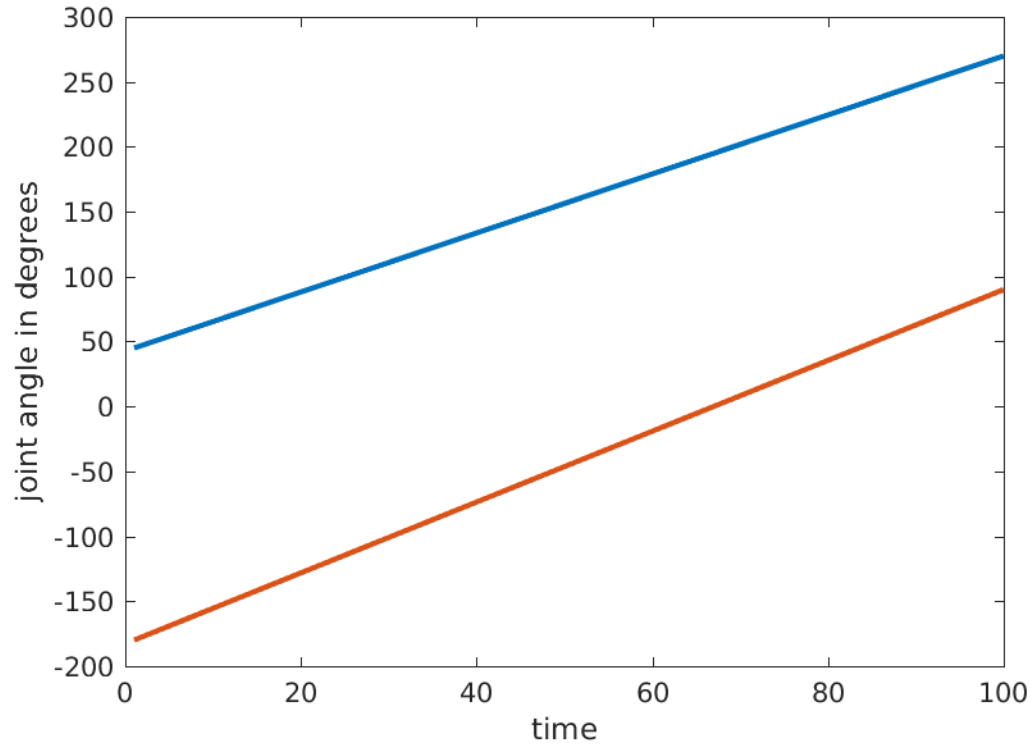
link 1

link 2

end effector path

Joint-Space Path Joint Angles

- ▶ linear joint-space path



Constraints

- ▶ in the previous example we had two constraints for joint 1:
 1. ${}^0\theta_1 = 60$
 2. ${}^f\theta_1 = 270$
- ▶ the simplest path satisfying these constraints is the straight line path
- ▶ if we add more constraints then a straight line path may not be able to satisfy all of the constraints

Velocity constraints

- ▶ a common constraint is that the robot starts from a stationary position and stops at a stationary positions
 - ▶ in other words, the joint velocities are zero at the start and end of the movement

$$3. \quad {}^0\left(\frac{d\theta_1}{dt}\right) = {}^0\dot{\theta}_1 = 0$$

$$4. \quad {}^f\left(\frac{d\theta_1}{dt}\right) = {}^f\dot{\theta}_1 = 0$$

- ▶ more generally, we might require non-zero velocities

$$3. \quad {}^0\left(\frac{d\theta_1}{dt}\right) = {}^0\dot{\theta}_1 = {}^0v$$

$$4. \quad {}^f\left(\frac{d\theta_1}{dt}\right) = {}^f\dot{\theta}_1 = {}^fv$$

Acceleration constraints

- ▶ for smooth motion, we might require that the acceleration at the start and end of the motion be zero

5.
$${}^0\left(\frac{d^2\theta_1}{dt^2}\right) = {}^0\ddot{\theta}_1 = 0$$

6.
$${}^f\left(\frac{d^2\theta_1}{dt^2}\right) = {}^f\ddot{\theta}_1 = 0$$

- ▶ more generally, we might require non-zero accelerations

5.
$${}^0\left(\frac{d^2\theta_1}{dt^2}\right) = {}^0\ddot{\theta}_1 = {}^0\alpha$$

6.
$${}^f\left(\frac{d^2\theta_1}{dt^2}\right) = {}^f\ddot{\theta}_1 = {}^f\alpha$$

Satisfying the constraints

- ▶ given some set of constraints on a joint variable q our goal is to find $q(t)$ that satisfies the constraints
- ▶ there are an infinite number of choices for $q(t)$
 - ▶ it is common to choose “simple” functions to represent $q(t)$

Satisfying the constraints with polynomials

- ▶ suppose that we choose $q(t)$ to be a polynomial
- ▶ if we have n constraints then we require a polynomial with n coefficients that can be chosen to satisfy the constraints
 - ▶ in other words, we require a polynomial of degree $(n - 1)$

Satisfying the constraints with polynomials

- ▶ suppose that we have joint value and joint velocity constraints

1. $q(t_0) = q_0$

2. $q(t_f) = q_f$

3. $\dot{q}(t_0) = v_0$

4. $\dot{q}(t_f) = v_f$

- ▶ we require a polynomial of degree 3 to represent $q(t)$

- ▶ $q(t) = a + bt + ct^2 + dt^3$

- ▶ the derivative of $q(t)$ is easy to compute

- ▶ $\dot{q}(t) = b + 2ct + 3dt^2$

Satisfying the constraints with polynomials

► equating $q(t)$ and $\dot{q}(t)$ to each of the constraints yields:

1. $q(t_0) = q_0 = a + bt_0 + ct_0^2 + dt_0^3$

2. $q(t_f) = q_f = a + bt_f + ct_f^2 + dt_f^3$

3. $\dot{q}(t_0) = v_0 = b + 2ct_0 + 3dt_0^2$

4. $\dot{q}(t_f) = v_f = b + 2ct_f + 3dt_f^2$

which is a linear system of 4 equations with 4 unknowns
(a, b, c, d)

Example

- ▶ consider the following constraints where the robot is stationary at the start and end of the movement

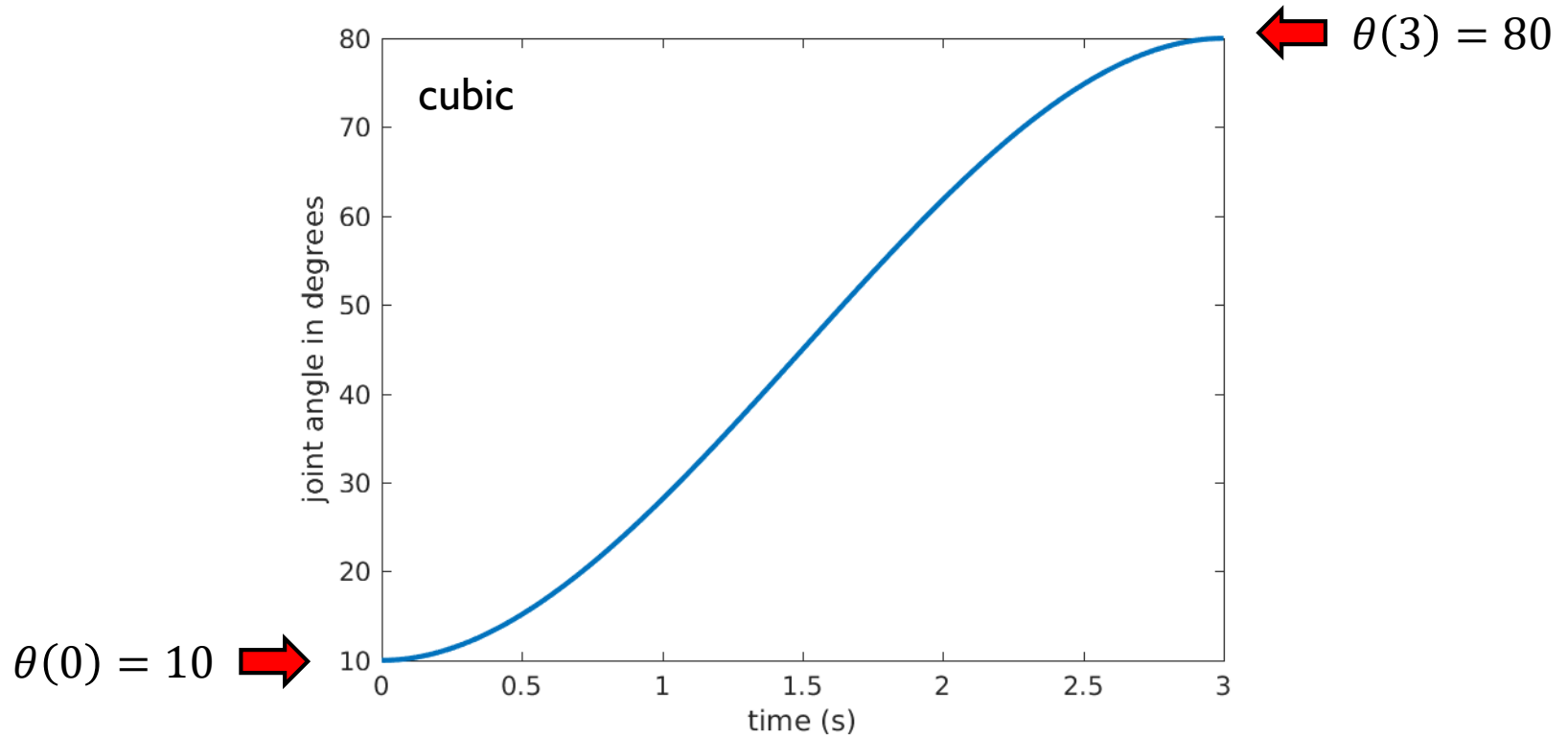
1. $q(t_0) = \theta(0) = 10$

2. $q(t_f) = \theta(3) = 80$

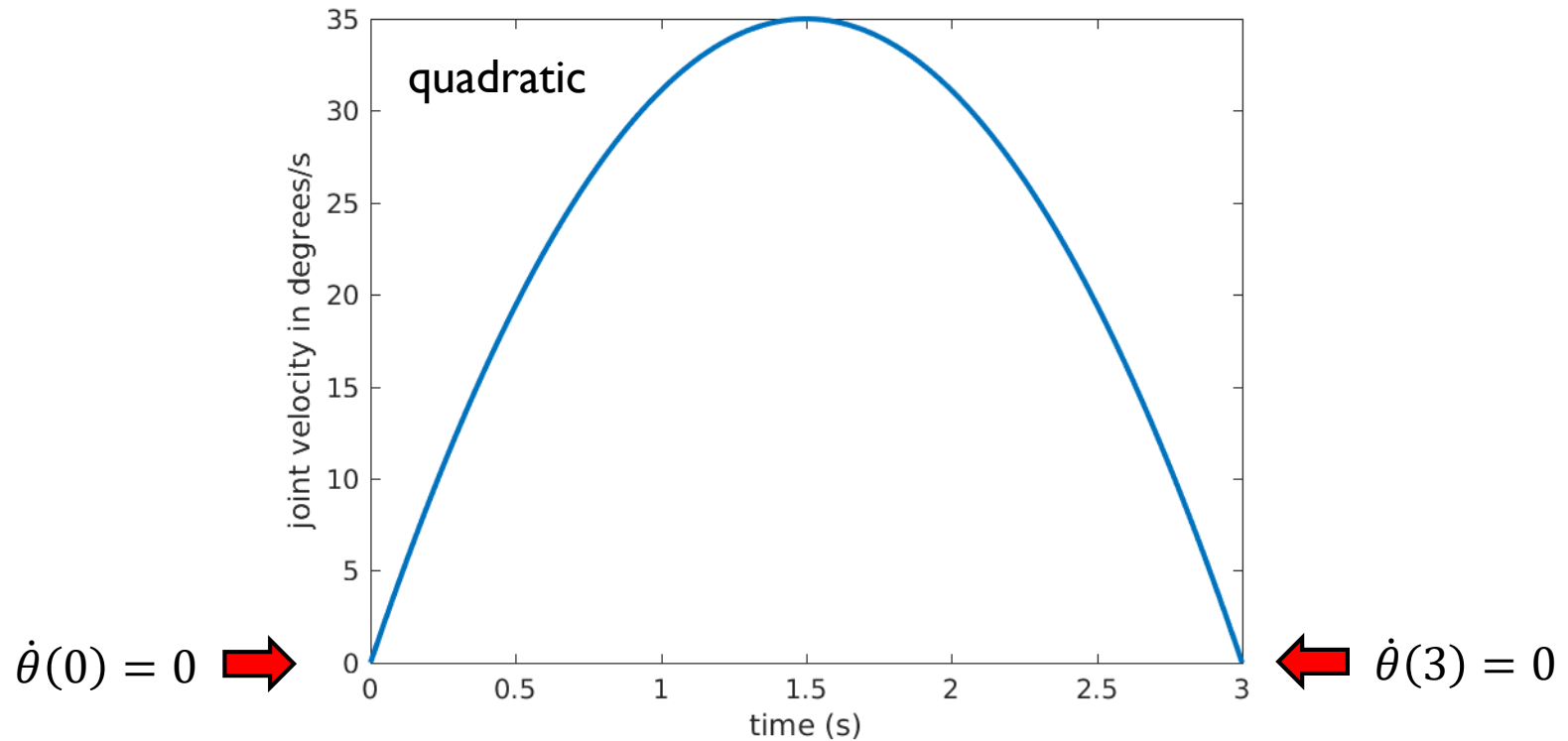
3. $\dot{q}(t_0) = \dot{\theta}(0) = 0$

4. $\dot{q}(t_f) = \dot{\theta}(3) = 0$

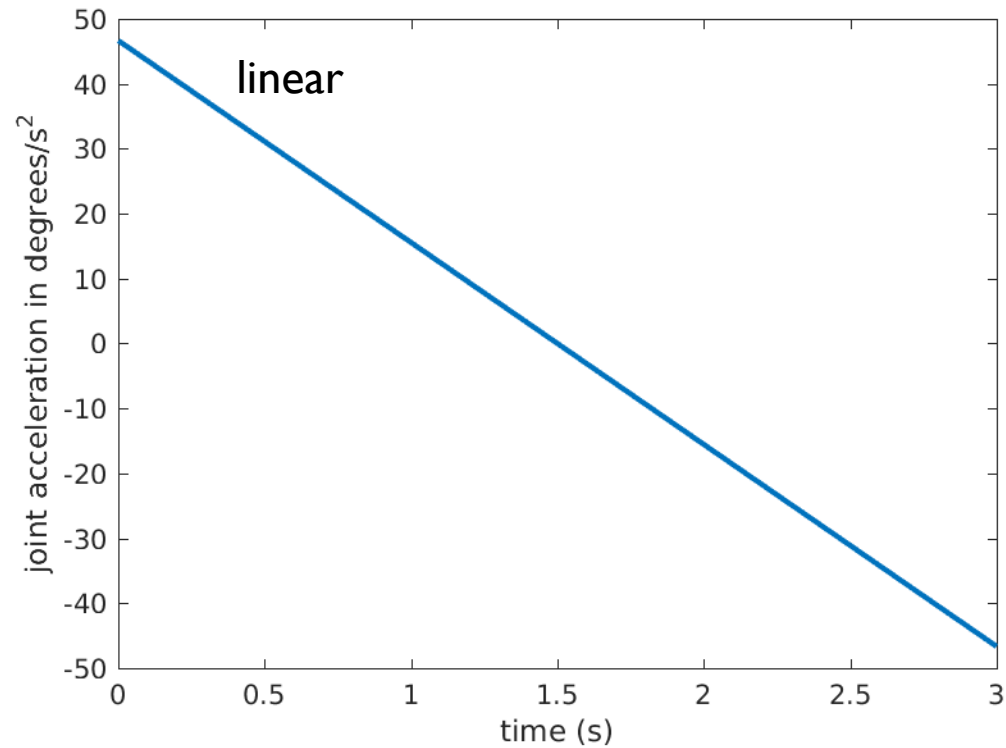
Example: Joint angle



Example: Joint velocity



Example: Joint acceleration



Satisfying the constraints with polynomials

- ▶ suppose that we have joint value, joint velocity, and joint acceleration constraints

1. $q(t_0) = q_0$

2. $q(t_f) = q_f$

3. $\dot{q}(t_0) = v_0$

4. $\dot{q}(t_f) = v_f$

5. $\ddot{q}(t_0) = \alpha_0$

6. $\ddot{q}(t_f) = \alpha_f$

Satisfying the constraints with polynomials

- ▶ we require a polynomial of degree 5 to represent $q(t)$

- ▶ $q(t) = a + bt + ct^2 + dt^3 + et^4 + ft^5$

- ▶ the derivatives of $q(t)$ are easy to compute

- ▶ $\dot{q}(t) = b + 2ct + 3dt^2 + 4et^3 + 5ft^4$

- ▶ $\ddot{q}(t) = 2c + 6dt + 12et^2 + 20ft^3$

Satisfying the constraints with polynomials

► equating $q(t)$, $\dot{q}(t)$, and $\ddot{q}(t)$ to each of the constraints yields:

1. $q(t_0) = q_0 = a + bt_0 + ct_0^2 + dt_0^3$

2. $q(t_f) = q_f = a + bt_f + ct_f^2 + dt_f^3$

3. $\dot{q}(t_0) = v_0 = b + 2ct_0 + 3dt_0^2$

4. $\dot{q}(t_f) = v_f = b + 2ct_f + 3dt_f^2$

5. $\ddot{q}(t_0) = \alpha_0 = 2c + 6dt_0 + 12et_0^2 + 20ft_0^3$

6. $\ddot{q}(t_f) = \alpha_f = 2c + 6dt_f + 12et_f^2 + 20ft_f^3$

which is a linear system of 6 equations with 6 unknowns
(a, b, c, d, e, f)

Example

- ▶ consider the following constraints where the robot is stationary at the start and end of the movement, and the joint accelerations are zero at the start and end of the movement

1. $q(t_0) = \theta(0) = 10$

2. $q(t_f) = \theta(3) = 80$

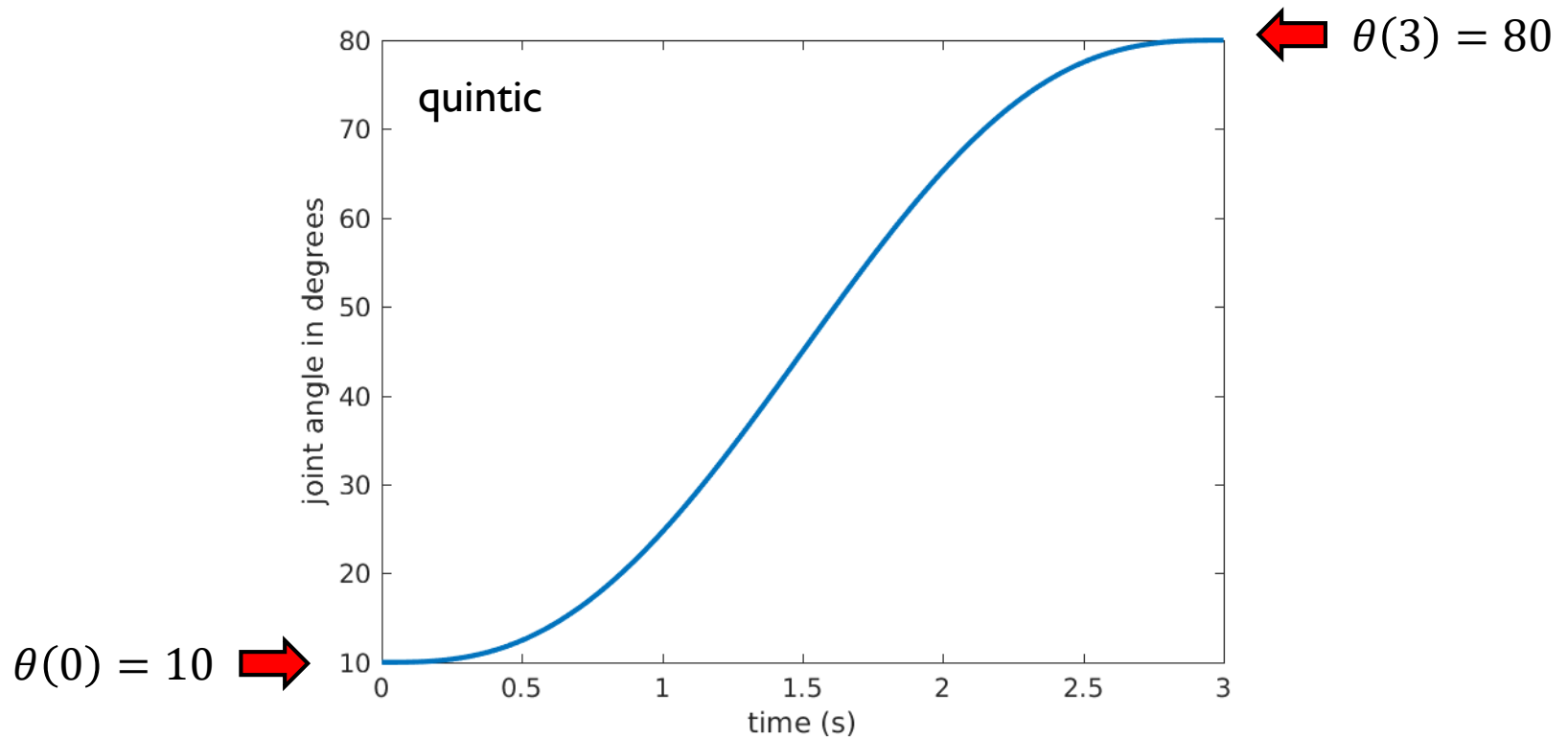
3. $\dot{q}(t_0) = \dot{\theta}(0) = 0$

4. $\dot{q}(t_f) = \dot{\theta}(3) = 0$

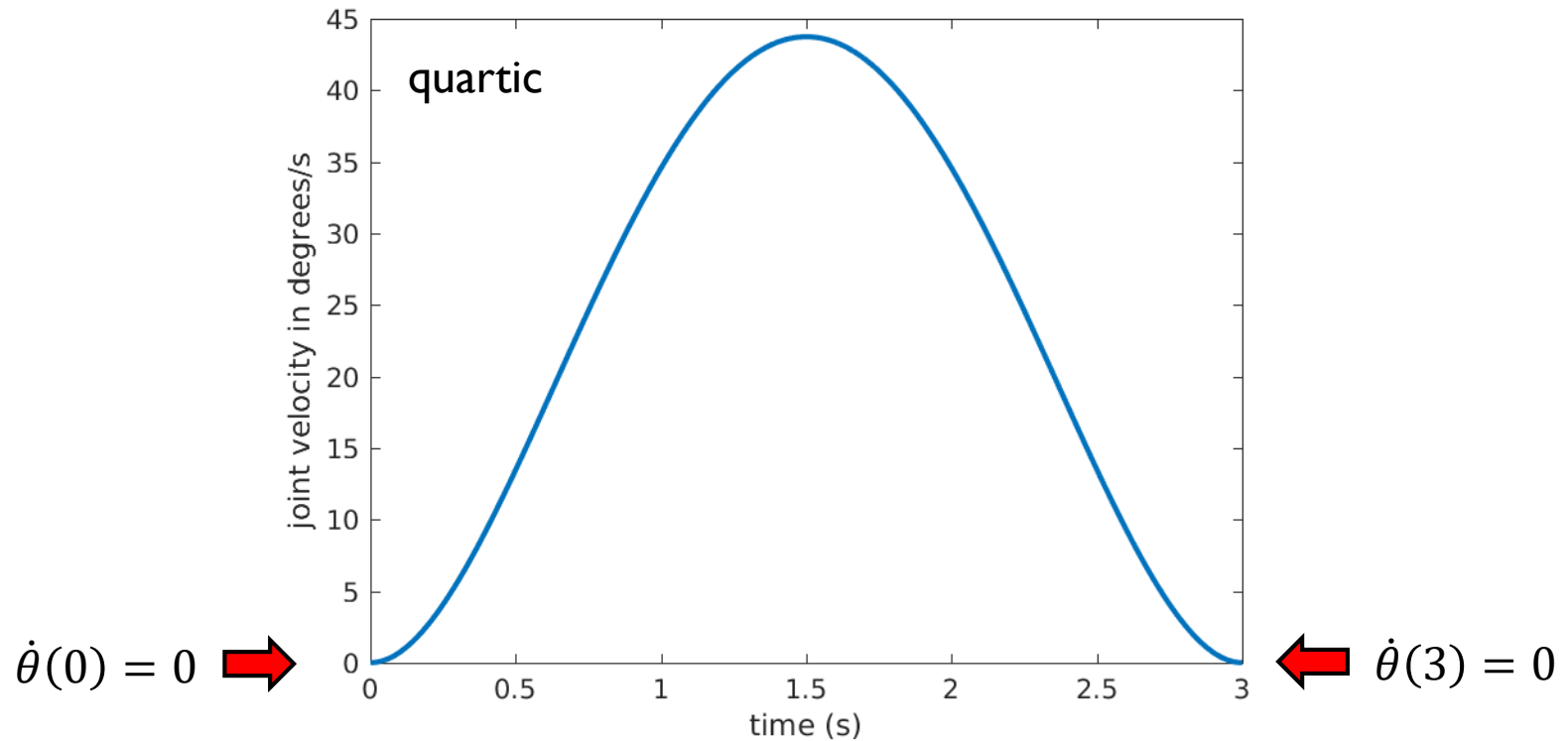
5. $\ddot{q}(t_0) = \ddot{\theta}(0) = 0$

6. $\ddot{q}(t_f) = \ddot{\theta}(3) = 0$

Example: Joint angle



Example: Joint velocity



Example: Joint acceleration

